MONTHLY NOTICES

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ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1938 MARCH 11

Dr. H. Spencer Jones, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Miss Madge Gertrude Adam, University Observatory, Oxford ;
Rev. George E. Harrison, The Manse, Smeaton, Victoria, Australia ;
Rev. Joseph Lynch, S.J., Dept. of Physics, Fordham University, New York City, N.Y., U.S.A. ; and
Lieut.-Col. J. E. Nicholson, The Gables, Somerset Road, Douglas, Isle of Man.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended :—

A. L. Narayan, M.A., D.Sc., Director of the Solar Physics Observatory, Kodaikanal, S. India (proposed by T. P. Bhaskaran).

One hundred and thirty-seven presents were announced as having been received since the last ordinary meeting, including :—

Bayer (Johannes), "Uranometria, omnium asterismorum continens schemata, nova methodo delineata, æreis laminis expressa" (1st edition), 1603 (presented by Mr. A. Stanley Williams) ;
Fauth (Philipp), "Unser Mond" (presented by Mrs. Evershed) ;
MacPike (E. F.), "Hevelius, Flamsteed and Halley" (presented by the author).

The President announced that, in order to meet the needs of the large body of Fellows who cannot read with profit some of the more technical

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papers in the *Monthly Notices*, the Council had decided to issue a Leaflet from time to time. The Leaflet would contain accurate astronomical information in a popular form on subjects of topical and non-topical interest, as well as notes of general interest about the affairs of the Society. The first issue would appear in June.

The President announced the death of Dr. George Ellery Hale, Fellow, Associate and Gold Medallist of the Society, who at the time of his death was the Associate of longest standing in the Society, having been elected an Associate in 1899. The Meeting stood in silence as a tribute of respect to Dr. Hale's memory.

A PARABOLIC DOUBLE-STAR ORBIT.

R. v. d. R. Woolley.

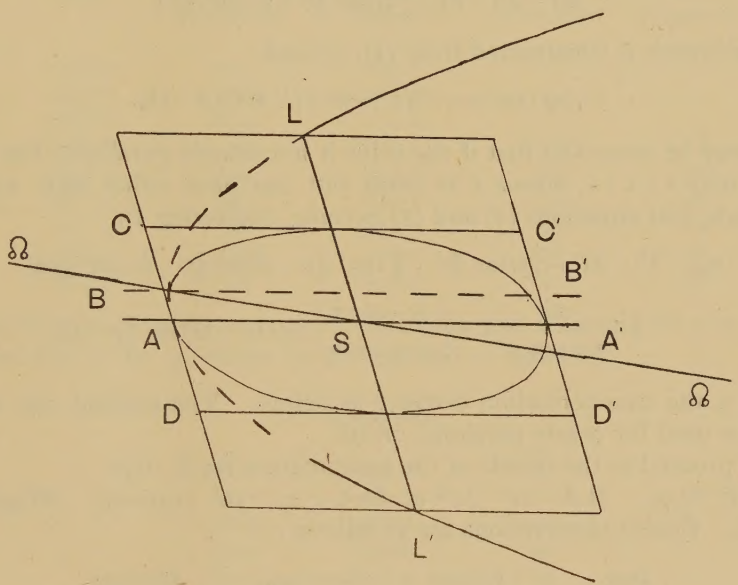
On attempting to represent the observations of position angle and separation of the double star ADS $8539 = \Sigma\ 1639$ by an elliptical orbit in the usual manner some difficulty was experienced, and as it was apparent that the eccentricity must be nearly equal to unity it was decided to attempt to account for the observations by a parabolic orbit.* This attempt met with success. The parabola was found by an appropriate modification of Russell's graphical method † as follows.

The method will only work if a substantial part of the orbit before periastron and another part after periastron have been reasonably well defined by the observations. The distance s and position angle p are plotted against the time and interpolation curves drawn and adjusted to make s^2dp/dt constant. Points are read off these curves, and rectangular co-ordinates $x = s\cos p$ and $y = s\sin p$ are plotted. These points lie on a conic section, the apparent orbit, which is an ellipse if the relative orbit (called the true orbit hereafter) is an ellipse, and a parabola if the true orbit is a parabola. In Russell's method a series of parallel chords is drawn, and the best straight line through their mid-points defines a diameter of the ellipse. In the parabolic case the same procedure defines a straight line parallel to the axis of the apparent parabola. Now erect a new series of chords perpendicular to this line : the mid-points of these chords define the axis of the apparent parabola. If the lengths of these chords are η_1, η_2, etc., and the ordinates measured along the axis from any point in it are ξ_1, ξ_2, etc., the apparent orbit is a parabola if $\eta^2 \propto \xi + a$, where a is constant. We assume that this relation is satisfied. If the orbit round periastron is insufficiently defined by the observations, it may be supplied by means of this relation.

* It was remarked by Jackson in the *Greenwich Catalogue of Double Stars* that a parabolic orbit would fit the observations.

† *M.N.*, **93**, 599, 1933.

The axis of the apparent orbit will not in general pass through the primary star. In the diagram BB′ is the axis of the apparent orbit and S the primary star. Through S draw SA parallel to BB′ to cut the orbit in A. Then A is the periastron point and SA is the projection of the axis of the true orbit. Through S draw a chord LL′ such that LS = SL′. Then LL′ is the projection of the latus rectum of the true orbit, and SA, SL are the projections of two lines in the plane of the true orbit which are perpendicular to one another and whose lengths are in the ratio 1 : 2. The parallelogram CC′D′D is the projection of a square in the plane of the true orbit. Draw a réseau in this parallelogram and plot on it the ordinates and abscissæ of a circle drawn on a rectangular réseau. The resulting figure is an ellipse,



as shown in the diagram. The major axis of this ellipse is the line of nodes of the orbit, and the cosine of the inclination is the ratio of the minor axis to the major axis of this ellipse.

In the usual notation let Ω be the position angle of the line of nodes and ω be the angle between the line of nodes and the axis of the parabola. If p_1 and p_2 are the position angles of SA and SL,

$$2 \operatorname{cosec} 2\omega = +\tan (p_1 - p_2) \sin i \tan i,$$
$$\tan (p_1 - \Omega) = \tan \omega \cos i,$$

whence ω and Ω, if we take $\cos i$ from the ratio of the axes of the ellipse. The value of Ω may be compared with the graphical value.

When i, ω and Ω have been determined, the true anomaly v corresponding to any position angle p is given by

$$\tan (v + \omega) = \tan (p - \Omega) \sec i. \qquad (1)$$

Finally, the true periastron distance q is given from the measured distance SA by

$$q = \mathrm{SA} \sec \omega \cos (p_1 - \Omega).$$

The connection with the time is through Barker's relation

$$t - T \propto \tan \tfrac{1}{2}v + \tfrac{1}{3} \tan^3 \tfrac{1}{2}v.$$

There is an extensive table of

$$M = 75(\tan \tfrac{1}{2}v + \tfrac{1}{3} \tan^3 \tfrac{1}{2}v)$$

in Klinkerfues' *Theoretische Astronomie*, so that a convenient relation to use is

$$n(t - T) = M = 75(\tan \tfrac{1}{2}v + \tfrac{1}{3} \tan^3 \tfrac{1}{2}v). \qquad (2)$$

The ephemeris is constructed from (1), (2) and

$$s = 2q \cos (v + \omega)(1 + \cos v)^{-1} \sec (p - \Omega). \qquad (3)$$

It may be remarked that if the orbit is not exactly parabolic, but has an eccentricity $e = 1 + \epsilon$, where ϵ is small but can have either sign, equation (1) stands, but equations (2) and (3) become, neglecting ϵ^2,

$$n(t - T) = M' = 75\{\tan \tfrac{1}{2}v + \tfrac{1}{3} \tan^3 \tfrac{1}{2}v - \epsilon(\tan \tfrac{1}{2}v - \tfrac{1}{5} \tan^5 \tfrac{1}{2}v)\}, \qquad (2')$$

$$s = \frac{2q}{1 + \cos v}\left(1 + \frac{\epsilon}{2}\frac{1 - \cos v}{1 + \cos v}\right) \cos (v + \omega) \sec (p - \Omega). \qquad (3')$$

Here q is the true perihelion distance as before. The method can, accordingly, be used for nearly parabolic orbits.

We proceed to the details of the computation for Σ 1639.

ADS 8539. R.A. $12^h\ 21^m{\cdot}9$, Dec. $+25°\ 52'$ (1950·0). Magnitudes 6·7, 7·9. Recent observations are as follows :—

Date	Position	Separation	Observer
	°	″	
1926·34	339·6	1·32	Giacobini
28·23	337·1	0·86	Pulkova
29·33	337·0	0·70	Pulkova
30·18	337·5	0·80	Pulkova
30·30	336·1	0·84	Kuiper
30·31	329·5	0·95	Warner
30·34	335·6	1·19	Doberck
31·32	336·6	0·81	Pulkova
32·19	336·7	0·84	Simonow
32·47	334·5	0·96	Finsen
33·35	333·9	1·01	Baize
34·46	334·8	0·88	Aitken
36·28	333·5	0·90	Duruy
36·42	333·1	1·11	Baize
37·33	333·2	1·06	Greenwich
37·47	332·3	0·91	Phillips
38·23	333·8	0·99	Phillips
38·26	333·8	1·04	Symms
38·26	333·9	1·14	Woolley

After a little empirical adjustment to the elements found graphically, the following set of elements was adopted :—

q	0″·100
i	126·1
☊	300°
ω	342°
n	26·3
T	1887·0

Values of the position angle and separation deduced from these elements are compared below with nine normal places.

Date	Position Angle		Separation	
	Observed	Computed	Observed	Computed
	°	°	″	″
1836·0	291	291·8	...	...
1858·0	286½	287·1	...	...
1876·0	274	274·2	...	...
1898·0	6½	6·1	0·29	0·28
1903·0	354	354·8	·39	·38
1910·0	347	346·5	·52	·53
1917·0	342	341·9	·67	·67
1923·0	339½	339·2	·78	·78
1933·0	335	336·2	·95	·96

The agreement is good in both co-ordinates, but it should be emphasized that the orbit is by no means to be regarded as definitive.

Ephemeris

	°	″
1940·0	334·5	1·08
45·0	333·6	1·16
50·0	332·7	1·24
55·0	332·0	1·32
60·0	331·4	1·39

The object should be followed carefully for a few years in the hope of determining the eccentricity.

There is one other pair to which a parabolic orbit has been fitted in recent times—to wit, ADS 7203, σ_2 UMa—for which Halpern * computed a parabolic orbit as well as an elliptical one of period 10850 years and eccentricity 0·96. It is, of course, possible that both orbits σ_2 UMa and Σ 1639 are weak hyperbolæ, and that we have here cases of stellar encounters. It would be of great interest to establish the hyperbolic nature of one or other of these orbits, if true, by an accumulation of observational material in the next decade.

Summary.—A graphical method of computing a parabolic double-star orbit is described and applied to Σ 1639. It is suggested that observations of this star and of σ_2UMa, to which a parabolic orbit has also been fitted, should be intensified in the hope of determining the eccentricities definitively.

* *A.J.*, **42**, 32, 1932.

THE DISTANCES OF THE EXTRA-GALACTIC NEBULÆ.

G. C. McVittie.

1. *Introduction*

The counts of extra-galactic nebulæ * published by the Mount Wilson Observatory have been interpreted by means of the expanding universes of general relativity in two different ways. On the one hand, Dr. E. Hubble † has suggested that they lead to a small closed universe filled with matter of high average density, whilst on the other hand the present author ‡ has contended that they indicate an infinite hyperbolic universe containing matter of low density. The discrepancy does not arise either from differences in method in performing the necessary extrapolations of the empirical formulæ or from a neglect of the spread in absolute magnitude of the observed nebulæ. The object of the present paper is to show that the essential difference between the two interpretations lies in the different definitions of distance adopted.

By the term "distance" we shall mean the result of a measurement by means of rigid scales, this definition being the fundamental one in general relativity. The distance of a spiral nebula will then be the distance so determined at the moment when the nebula is being observed. Distance in this sense is of course not directly measurable, but a measurement of apparent photographic magnitude together with the theoretical considerations of general relativity enable us to calculate it. In fact, it can be shown § that, provided the curvature of space be not too great, the distance l (in parsecs) of a nebula of observed apparent magnitude m, absolute magnitude M and exhibiting a red-shift || of amount δ is

$$\log_{10} l = 0{\cdot}2(m - \eta\delta - M) + 1, \qquad (1.1)$$

where η is a constant whose value general relativity does not give completely. The problem of determining the distances of the extra-galactic nebulæ resolves itself very largely into that of the value of η. Bound up with this problem is that of the curvature of space and of the structure of the universe as a whole.

In the following pages we shall analyse the various meanings which Hubble gives to the term distance in the observational formulæ on the one hand and in his model universe on the other. We shall find that his interpretation of the nebular counts depends on the hypothesis that η is

* *Ap. J.*, **84**, 517, 1936.

† E. Hubble, *Ap. J.*, **84**, 517, 1936.

‡ G. C. McVittie, *Cosmological Theory*, chap. iv, Methuen, 1937; *Zeits. f. Astrophysik*, **14**, 4, 274, 1937.

§ See, *e.g.*, § 4 below.

|| The connection between δ and the wave-length λ of the light emitted by the nebula is $\delta = d\lambda/\lambda$.

exactly equal to 4. A similar analysis applied to the author's method yields for η the value $3{\cdot}25 \pm 0{\cdot}10$. The observational formulæ suggest *a priori* that the value of η should be 3.

2. *Observational Formulæ and Distance*

The empirical formulæ between the number of nebulæ per square degree, N_m, and apparent photographic magnitude m, are deduced from the observations in the following form :—

$$\log_{10} N_m = 0{\cdot}6(m - \Delta) - 9{\cdot}052 \pm 0{\cdot}013, \tag{2.1}$$

$$\log_{10} \Delta \quad = 0{\cdot}2(m - \Delta) - 4{\cdot}239 \pm 0{\cdot}024, \tag{2.2}$$

where Δ is interpreted as being a "correction" to m. No use whatever is made of distance in the actual derivation * of these formulæ which depend only on the observed values of N_m and m, and on the assumption of certain functional forms for the dependence of N_m and Δ on m. If we choose *a posteriori* to *interpret* (2.1) as implying a *uniform distance distribution* of nebulæ, we must define "distance" by means of the formula

$$\log_{10} D = 0{\cdot}2(m - \Delta - M) + 1, \tag{2.3}$$

where M is the absolute magnitude of the nebulæ. In this connection it may be remarked that Hubble has put forward the view that the red-shift would then be due to velocity if the observational value of Δ turned out to be 4δ, but would arise from some unexplained physical mechanism if Δ was found to be 3δ. In terms of D (2.1) becomes

$$N_m = 10^{0{\cdot}6M - 12{\cdot}052 \pm 0{\cdot}013} D^3. \tag{2.4}$$

The empirical relations between the observed red-shift δ and the observed apparent photographic magnitude m of the nebulæ, which we shall call "displacement-magnitude" relations, have been given in the following forms. Up to 1934 we had for the field-nebulæ †

$$\log_{10} \delta_0 = 0{\cdot}2m - 4{\cdot}967 \pm 0{\cdot}012. \tag{2.5}$$

More recently the quantity 3δ has been arbitrarily subtracted from the observed m of every nebula whose δ has been measured. The resulting displacement-magnitude relation for field-nebulæ takes the form ‡

$$\log_{10} \delta_1 = 0{\cdot}2(m - 3\delta_1) - 4{\cdot}707 \pm 0{\cdot}016, \tag{2.6}$$

whilst that for cluster-nebulæ is

$$\log_{10} \delta_2 = 0{\cdot}2(m - 3\delta_2) - 4{\cdot}457. \tag{2.7}$$

Each of these formulæ has been interpreted as a *linear* relation between δ

* E. Hubble, pp. 530 *et seq.*

† E. Hubble and M. L. Humason, *Proc. Nat. Acad. Sci.*, **20**, 264, 1934. We denote by δ_0 the value of δ for given m calculated from this formula in order to distinguish it from the values calculated from (2.6) and (2.7) respectively.

‡ E. Hubble, *Ap. J.*, **84**, 270, 1936.

and "distance." * We have therefore here three more definitions of "distance," associated with (2.5), (2.6) and (2.7) respectively, viz.

$$\log_{10} D_0 = 0{\cdot}2(m - M_0) + 1 \qquad (M_0 = -14{\cdot}9), \tag{2.8}$$

$$\log_{10} D_1 = 0{\cdot}2(m - 3\delta_1 - M_1) + 1 \qquad (M_1 = -15{\cdot}15),\dagger \tag{2.9}$$

$$\log_{10} D_2 = 0{\cdot}2(m - 3\delta_2 - M_2) + 1 \qquad (M_2 = -16{\cdot}45).\dagger \tag{2.10}$$

We remark that the "distances" D, D_0, D_1, D_2 have all been suggested by the observational formulæ themselves with very little appeal to theory. In Hubble's work, however, they are all discarded in favour of two quantities, which we shall call D_3 and D_4, defined as follows :—

$$\log_{10} D_3 = 0{\cdot}2(m - 4\delta_1 - M_1) + 1, \tag{2.11}$$

which applies to field-nebulæ, and

$$\log_{10} D_4 = 0{\cdot}2(m - 4\delta_2 - M_2) + 1,$$

applying to clusters. These two "distances" ‡ are regarded as being "true" and different from the others.

We now consider the general dependence between δ and a distance-variable r where

$$\log_{10} \delta = 0{\cdot}2(m - 3\delta) - X, \tag{2.12}$$

$$\log_{10} r = 0{\cdot}2(m - \eta\delta) + Q, \tag{2.13}$$

X, η and Q being constants. If we eliminate m from these formulæ we have

$$10^{0{\cdot}2(3-\eta)\delta}.\delta = 10^{-(X+Q)}.r.$$

Solving this equation for δ to the second approximation there comes

$$\delta = 10^{-(X+Q)}r + 0{\cdot}2E(\eta - 3)10^{-2(X+Q)}r^2, \tag{2.14}$$

where

$$E = \log_e 10 = 2{\cdot}303.$$

To obtain the connection § between δ and D_3, we identify δ with δ_1 and r with D_3 by writing

$$X = 4{\cdot}707, \qquad \eta = 4, \qquad Q = 1 - 0{\cdot}2M_1 = 4{\cdot}03,$$

so that (2.14) becomes

$$\delta_1 = 1{\cdot}83 \times 10^{-9}D_3 + 1{\cdot}54 \times 10^{-18}D_3{}^2. \tag{2.15}$$

It is important to observe that the coefficient of r in (2.14) is independent of η, but that the coefficient of r^2 is very sensitive to the value adopted for

* E. Hubble, *Ap. J.*, **84**, 533, 544, 1936.

† E. Hubble, *Proc. Nat. Acad. Sci.*, **22**, 621, note 8, 1936.

‡ E. Hubble, *Ap. J.*, **84**, 545, footnote 30, 1936.

§ E. Hubble (*loc. cit.*, p. 545), apparently using graphical methods, gives $\delta_1 = 1{\cdot}75 \times 10^{-9}D_3 + 2{\cdot}7 \times 10^{-18}D_3{}^2$.

this number. Obviously, if r is identified with D_1 or D_2, so that $\eta = 3$, the coefficient of r^2 is zero. Whereas therefore (2.5), (2.6) and (2.7) can be regarded with confidence as being empirical formulæ, the relation (2.15) is of value only if it can be conclusively proved that D_3 is of physical significance. In any case we are hardly entitled to follow Hubble in calling this formula "empirical."

The connection between Δ and δ, as expressed by Hubble,* involves an appeal to "distance" in order to justify the existence of an approximately linear relation between these quantities. Such a relation is, however, obvious by inspection of (2.2), (2.5), (2.6) and (2.7). Eliminating m from these equations, we obtain

$$\Delta = 5{\cdot}35\delta_0, \tag{2.16}$$

$$\Delta = 2{\cdot}94\delta_1, \tag{2.17}$$

$$\Delta = 1{\cdot}65\delta_2. \tag{2.18}$$

Of these three possibilities, Hubble rejects the first and last and retains (2.17) apparently on the ground that, by writing $\Delta = 2{\cdot}94\delta_1$ and $M = M_1$ in (2.3), the definitions of D and D_1 then become identical. If the nebular counts were supposed to refer to field-nebulæ alone there might be some justification for this. But in calculating the curvature of space, (2.1) is treated as if it gave *all* the observed nebulæ inclusive of those in clusters. With more plausibility it might be argued that since the displacement-magnitude relation for clusters refers to the greater depths of space and therefore to regions more nearly comparable with those involved in the nebular counts, (2.18) ought to be regarded as the correct result.

A linear connection between Δ and δ therefore indicates merely that the definition (2.3) of D is

$$\log_{10} D = 0{\cdot}2(m - a\delta - M) + 1, \tag{2.19}$$

where a is a constant whose value lies between 5·35 and 1·65. Moreover, this constant is primarily a function of the particular displacement-magnitude relation selected in the determination of Δ as a function of δ. Its value is therefore largely at the disposal of the investigator and so can have little physical significance. This conclusion throws serious doubts on the possibility of discriminating observationally between different "causes" for the red-shift in the manner suggested by Hubble, the proposed test being that the red-shift would be due to velocity if a turned out to have the value 4, but due to some other cause if a had happened to be 3.

There is, of course, no harm in selecting the value 2·94 for a provided it be realised that this choice is largely arbitrary. Equivalently, we are choosing D to be identical with either D_1 or D_2, according to our choice of M in (2.19), and thus reducing the number of different definitions of "distance" by one. At the same time we are suggesting that the observational value of η in (1.1) is 3.

* E. Hubble, *loc. cit.*, p. 533.

3. *The Model Universe*

When interpreting the nebular counts by means of the model universes of general relativity using Hubble's method, the following formulæ are employed. Firstly, using (2.1) with (2.17) and dropping the suffix of δ, we have

$$\log_{10} N_m = 0{\cdot}6(m - 2{\cdot}94\delta) - 9{\cdot}052 \pm 0{\cdot}013, \tag{3.1}$$

where δ is given by the displacement-magnitude relation for field-nebulæ, viz.

$$\log_{10} \delta = 0{\cdot}2(m - 3\delta) - 4{\cdot}707 \pm 0{\cdot}016. \tag{3.2}$$

"Distance" is given by

$$\log_{10} D_3 = 0{\cdot}2(m - 4\delta - M_1) + 1, \tag{3.3}$$

where $M_1 = -15{\cdot}15$. Alternatively to (3.2), the form

$$\delta = 1{\cdot}75 \times 10^{-9} D_3 + 2{\cdot}7 \times 10^{-18} D_3^2 \tag{3.4}$$

is employed for the determination of certain features of the model. We may also use the formula, obtained from (2.4) by putting $M = M_1$, and equivalent to (3.1), viz.

$$N_m = 10^{0{\cdot}6M_1 - 12{\cdot}052} D^3, \tag{3.5}$$

where

$$\log_{10} D = 0{\cdot}2(m - 2{\cdot}94\delta - M_1) + 1. \tag{3.6}$$

(i) *Distance.*—The model universe has the metric

$$ds^2 = dt^2 - e^{g(t)}\left(\frac{dr^2}{1 - kr^2/R_0^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{3.7}$$

where $k = +1$, -1 or 0. We consider an observer O at the spatial origin of co-ordinates $r = 0$, $\theta = 0$, $\phi = 0$, who is making his observations at the instant t_0. The function $g(t)$ can be selected so that $g(t_0) = 0$, and R_0 is therefore the value of the radius of curvature of space at the instant t_0. The co-ordinates r, t we assume to be measured in parsec so that unit t corresponds to $(3{\cdot}08/3) \times 10^8$ seconds. Consider now a light-source A having fixed co-ordinates (r, θ, ϕ) which emits a light-ray at the instant t_1, this light-ray being observed by O at t_0. We must first decide on what we are to mean by the distance of A from O. If we adopt the principles of general relativity, we mean by this distance the result of a measurement with rigid scales. At the instant t_0 the distance of A from O, as determined by the latter, is therefore

$$l = \int_0^r \frac{dr}{(1 - kr^2/R_0^2)^{\frac{1}{2}}} = r\left(1 + \frac{k}{6}\frac{r^2}{R_0^2} + \ldots\right). \tag{3.8}$$

It is immediately clear that the co-ordinate r represents distance only in those regions of the model where r^2/R_0^2 may be regarded as negligible. We shall see below (eq. (3.23)) that, in the model which Hubble uses, the quantity r^2/R_0^2 is of the order of δ. We cannot therefore assume that r

and l are identical in this model as is done by Hubble.* Particularly is this so since we are trying to decide which of the quantities D to D_4 is to be regarded as distance in the above exact sense, and these quantities differ essentially from one another by terms of the order of δ.

In the actual universe, of course, a direct distance-measurement by means of rigid scales is not possible in the case of a spiral nebula. Estimates based on measurements of apparent magnitude have their counterpart in the model in the determination of the apparent *bolometric* magnitude of A by the observer O. This gives the "luminosity-distance" L where

$$L = r(1+\delta). \tag{3.9}$$

Here δ is the red-shift in the light emitted by A and is defined by

$$\delta = e^{-\frac{1}{R}g(t_1)} - 1. \tag{3.10}$$

We note again that the co-ordinate r cannot be regarded as luminosity-distance. If we denote by K the correction depending on the ratio of the bolometric to the photographic luminosity of A, it can be shown † that the relationship between the co-ordinate r, the apparent photographic m and the absolute photographic magnitude M of A, is

$$\log_{10} r = 0{\cdot}2\{m - 5\log_{10}(1+\delta) - K - M\} + 1,$$

or, neglecting powers of δ higher than the first,

$$\log_{10} r = 0{\cdot}2\{m - (2{\cdot}17\delta + K) - M\} + 1. \tag{3.11}$$

Assuming for the moment that K is approximately a linear function of δ, we introduce the constant η and write

$$\left.\begin{aligned} \eta\delta &= 2{\cdot}17\delta + K \\ Q &= 1 - 0{\cdot}2M \end{aligned}\right\}, \tag{3.12}$$

and obtain

$$\log_{10} r = 0{\cdot}2(m - \eta\delta) + Q \tag{3.13}$$

in conformity with (2.13). The theory of relativity gives us no information whatever about the value of K, which depends on the distribution of energy in the spectrum of the light-source A and the distortion produced in this distribution by the red-shift. *A priori* estimates of K may therefore be arrived at by assuming that the spectrum of A is similar to that of an average spiral nebula of the actual universe. The distribution of energy in nebular spectra is not known with any degree of accuracy,‡ but these spectra present some general resemblances to that of the Sun. Taking this as his starting-point, Hubble assumes that K may be found by regarding the undistorted distribution to be that of black-body radiation of temperature $T = 6000°$. Admittedly this introduces § still further uncertainties. We should therefore expect the value of K to be subject to a large probable error. Hubble,

* See, e.g., *Proc. Nat. Acad. Sci.*, **22**, 625, 1936.
† E. Hubble and R. C. Tolman, *Ap. J.*, **82**, 308 *et seq.*, 1935.
‡ E. Hubble, *Ap. J.*, **84**, 535, 1936.
§ E. Hubble, *loc. cit.*, p. 538.

however, believes that K is known with such a high degree of accuracy in terms of δ that we may write

$$K = 1{\cdot}83\delta \tag{3.14}$$

and *regard the error in the coefficient* 1·83 *as altogether negligible compared with unity.* Hence by (3.12) η is exactly equal to 4. If also we write $M = M_1$, (3.13) becomes

$$\log_{10} r = 0{\cdot}2(m - 4\delta - M_1) + 1, \tag{3.15}$$

which is the form adopted by Hubble for the relationship between the co-ordinate r and the apparent photographic magnitude of a light-source A in his model universe.

It follows from (3.3) and (3.15) that, if we identify A with an average spiral nebula of the actual universe, then r is also identical with D_3. But since r is not the same thing as distance in the model, this does not endow the quantity D_3 with the significance of physical distance which in the model is l. We have merely succeeded in identifying D_3 with an arbitrarily chosen radial co-ordinate in the model, this identification depending on the hypothesis that (3.14) is strictly accurate.*

(ii) *Distribution of Nebulæ.*—The model universe (3.6) is by definition filled with a distribution of matter which, around every point (r, θ, ϕ), is strictly homogeneous and isotropic. If the matter be regarded as constituted by particles with fixed co-ordinates (r, θ, ϕ), these particles must clearly be observationally indistinguishable from one another. They cannot differ either in their masses or in their absolute magnitudes. The number of such particles lying in a shell of radii r and $r + dr$, visible simultaneously to O, is

$$dN = \frac{4\pi\alpha}{R_0^3} \frac{r^2 dr}{(1 - kr^2/R_0^2)^{\frac{1}{2}}}, \tag{3.16}$$

where α is a constant pure number. Integrating we obtain, for a sphere of radius r centred at O,

$$N = \frac{4\pi\alpha}{3} \frac{r^3}{R_0^3} P(r, k) = \frac{4}{3}\pi\alpha \frac{r^3}{R_0^3}\left(1 + \frac{3k}{10}\frac{r^2}{R_0^2} + \ldots\right). \tag{3.17}$$

The exact expressions for $P(r, k)$ are

$$P(r, +1) = \frac{3}{2}\frac{R_0^3}{r^3}\left\{\sin^{-1}\frac{r}{R_0} - \frac{r}{R_0}\left(1 - \frac{r^2}{R_0^2}\right)^{\frac{1}{2}}\right\}, \tag{3.18}$$

$$P(r, 0) = 1,$$

$$P(r, -1) = \frac{3}{2}\frac{R_0^3}{r^3}\left\{\frac{r}{R_0}\left(1 + \frac{r^2}{R_0^2}\right)^{\frac{1}{2}} - \sinh^{-1}\frac{r}{R_0}\right\}. \tag{3.19}$$

The number N is related to the number N_m of particles visible per square degree to O by

$$N = 4{\cdot}1 \times 10^4 N_m.$$

* Similar remarks apply to D_4.

Hence (3.17) is

$$N_m = \frac{4\pi a}{12{\cdot}3 \times 10^4} \frac{r^3}{{R_0}^3} P(r, k). \tag{3.20}$$

We shall now identify these particles of the model with the spiral nebulæ of the actual universe, neglecting the fact that the nebulæ not only differ from one another by their masses and their absolute magnitudes, but also do not really form the homogeneous and isotropic distribution of matter in space postulated by the theory. We assume that a direct comparison between (3.5) and (3.20) is possible. Hence

$$\frac{4\pi a}{12{\cdot}3 \times 10^4 {R_0}^3} = 10^{0{\cdot}6M_1 - 12{\cdot}052}, \tag{3.21}$$

$$D = rP^{\frac{1}{3}}(r, k). \tag{3.22}$$

The latter equation determines R_0. It may be written

$$3 \log_{10} D - 3 \log_{10} r = \log_{10} P(r, k).$$

Writing $Q = 1 - 0{\cdot}2M_1$ in (3.13) and using (3.6), we obtain

$$(\eta - 2{\cdot}94)\delta = \frac{10}{6} \log_{10} P(r, k) = \frac{1}{2E}\left(k\frac{r^2}{{R_0}^2} + \ldots\right). \tag{3.23}$$

If now we write $\eta = 4$, *i.e.* $K = 1{\cdot}83\delta$, it follows that we must take $k = +1$ in (3.23) so that space is spherical. Moreover, (3.23) now becomes

$$1{\cdot}06\delta = \tfrac{10}{6}F(r), \tag{3.24}$$

where

$$F(r) = \log_{10} P(r, +1) = \log_{10}\left\{\frac{3}{2}\frac{{R_0}^3}{r^3}\left(\sin^{-1}\frac{r}{R_0} - \frac{r}{R_0}\left(1 - \frac{r^2}{{R_0}^2}\right)^{\frac{1}{2}}\right)\right\}.$$

This is Hubble's equation for R_0: to calculate this quantity we write (3.24) in the form

$$\epsilon = 1{\cdot}06\delta - \tfrac{10}{6}F(r), \tag{3.25}$$

and then find corresponding values of r and δ respectively from (3.2) and (3.15) throughout the range $m = 18{\cdot}47$ to $m = 21{\cdot}03$ for which the nebular counts were made. The calculations follow closely those made by Hubble.* It is impossible to find R_0 so that $\epsilon = 0$, but we may regard the equation (3.24) as satisfied if ϵ is of the order of $0{\cdot}075$. This gives $R_0 = 1{\cdot}45 \times 10^8$ parsec, as found by Hubble.

That the value of R_0 determined in this way is very sensitive to that of η, and therefore to that of K, is sufficiently obvious from equation (3.23). We shall, however, recalculate R_0 for $\eta = 3$, corresponding to † $K = 0{\cdot}83\delta$.

* E. Hubble, *loc. cit.*, Table XI. In Hubble's notation $\Delta m_c = 4\delta$, $\Delta m_0 = 3\delta$, $D = 1{\cdot}06\delta$.

† Hubble's direct method of calculating K makes this value correspond to $T = 7500°$ (*Ap. J.*, **84**, 517, Table VI, 1936).

Instead of (3.25) we have approximately by (3.23)

$$\epsilon = 0{\cdot}06\delta - \frac{k}{4{\cdot}606}\frac{r^2}{R_0^2}. \tag{3.26}$$

Referring to (3.11) we have also now

$$\log_{10} r = 0{\cdot}2(m - 3\delta - M_1) + 1. \tag{3.27}$$

Since $M_1 = -15{\cdot}15$, (3.2) gives $\delta = 0{\cdot}230$ at the outer limit of the survey where $m = 21{\cdot}03$, and correspondingly we have, by (3.27), $r = 10^{8{\cdot}098}$ parsec. Hence the maximum value of ϵ is, by (3.26),

$$\epsilon = {\cdot}014 - \frac{k}{4{\cdot}606}\left(\frac{10^{8{\cdot}098}}{R_0}\right)^2. \tag{3.28}$$

Moreover, following Hubble, we regard (3.26) as satisfied whenever $|\epsilon| \leqslant {\cdot}075$. By (3.28) we may therefore give to k the values $+1$, -1 or 0 indifferently provided that

$$\frac{1}{4{\cdot}606}\left(\frac{10^{8{\cdot}098}}{R_0}\right)^2 \leqslant {\cdot}06.$$

This gives $R_0 \geqslant 2{\cdot}38 \times 10^8$ parsec. Thus, changing the value K from $1{\cdot}83\delta$ to $0{\cdot}83\delta$ has altered R_0 from the definite value of $1{\cdot}45 \times 10^8$ parsec to *any one* of the values lying between $2{\cdot}38 \times 10^8$ parsec and $+\infty$. At the same time the value of k has become indeterminate, and it is no longer possible to say whether space is spherical, hyperbolic or flat. It is clear, therefore, that the structure of the universe is made to depend almost entirely on K, *i.e.* on the distortion produced by the red-shift on the little-known energy distribution in nebular spectra.

Incidentally the equation (3.23) verifies our remark made earlier that in Hubble's model where $\eta = 4$ the quantity r^2/R_0^2 is of order δ, so that r and distance l are not identical.

(iii) *Density of Matter in Space.*—The density of matter in space predicted by Hubble's model may be calculated in two ways. We may either use the equation (3.21) for a direct determination, or we may employ Einstein's equations and use the value of R_0 already found. Calculating first by the direct method, we note that, at the instant t_0 in the model universe (3.7), the proper-volume within the sphere of radius r is

$$V = \int_0^r\int_0^\pi\int_0^{2\pi} \frac{r^2 \sin\theta dr d\theta d\phi}{(1 - kr^2/R_0^2)^{\frac{1}{2}}} = \frac{4\pi}{3} r^3 P(r, k).$$

Hence, using (3.17), we obtain

$$N = \frac{\alpha}{R_0^3} V.$$

The number of nebulæ per cubic parsec is therefore

$$\frac{N}{V} = \frac{\alpha}{R_0^3}.$$

The mass of an average nebula is estimated to lie between $2 \times 10^{9}\odot$ and $2 \times 10^{11}\odot$, so that we may take this mass to be $4 \times 10^{43\pm1}$ gr. The predicted average density of matter ρ is therefore

$$\rho = \frac{a}{R_0^3}\frac{4 \times 10^{43\pm1}}{(3{\cdot}08)^3 10^{54}} \text{ gr./cm.}^3.$$

Using now (3.21) with $M_1 = -15{\cdot}15$, we obtain

$$\rho = 9{\cdot}66 \times 10^{-30\pm1} \text{ gr./cm.}^3. \tag{3.29}$$

This result is in close agreement with the value of $8{\cdot}29 \times 10^{-30\pm1}$ gr./cm.^{3} obtained by the present author using a quite different method.*

Turning now to the determination by means of Einstein's equations, we write these for the metric (3.7) as

$$\kappa\rho = \frac{3k}{R_0^2} + \tfrac{3}{4}(g_0')^2 - \Lambda, \tag{3.30}$$

$$\kappa p = -\frac{k}{R_0^2} - g_0'' - \tfrac{3}{4}(g_0')^2 + \Lambda. \tag{3.31}$$

Here a dash denotes differentiation with respect to t, and the suffix zero implies that the derivatives are calculated at the instant t_0. The density ρ and the pressure p are measured in gr./cm.^{3} so that $\kappa = 8\pi G/c^2$, where G is the gravitational constant and $8\pi G = 1{\cdot}674 \times 10^{-6}$ c.g.s. units. Since R_0 is expressed in parsecs, c is in parsec per second. Λ is the cosmical constant. In order to calculate ρ and p we must first obtain g_0' and g_0'', which we do by establishing the theoretical equation corresponding to (3.4). We have (3.10) as well as the equation expressing the fact that the path of a light-ray is a null-geodesic of (3.7), viz.

$$-\int_{t_0}^{t_1} e^{-\frac{g}{2}} dt = \int_0^r \frac{dr}{(1 - kr^2/R_0^2)^{\frac{1}{2}}}.$$

We thus obtain by elimination of t_1 the equation

$$\delta = \frac{g_0'}{2} r + r^2\left(\frac{k}{12}\frac{r}{R_0^2} g_0' - \frac{g_0''}{4}\right), \tag{3.32}$$

which holds to the second approximation. Since we are dealing with Hubble's model we can identify r with D_3 and put $k = +1$. Comparing then the coefficient of r in (3.32) with that of D_3 in (3.4), we obtain

$$\tfrac{1}{2} g_0' = 1{\cdot}75 \times 10^{-9}.$$

Comparison of the "coefficient" of r^2 with that of ${D_3}^2$ is hardly legitimate as a means of finding g_0'', owing to the fact that the small value of R_0 makes $rg_0'/12R_0^2$ of the order of 10^{-18} throughout the region of the survey.

* G. C. McVittie, *Zeits. f. Astrophysik*, **14**, 4, 274, 1937.

However, it is sufficient to assume that $g_0''/4$ is of the order of 10^{-18}. With $R_0 = 1{\cdot}45 \times 10^8$ parsec, the terms in $(g_0')^2$ and g_0'' in Einstein's equations can be neglected compared with the rest. Assuming as usual that $p = 0$, we have $\Lambda = \frac{1}{R_0^2}$, and

$$\rho = \frac{2}{\kappa R_0^2} = 5{\cdot}4 \times 10^{-27} \text{ gr./cm.}^3, \tag{3·33}$$

which is approximately the value given by Hubble * for the density of matter in space. The important feature of the result (3·33) is that it has been obtained without a knowledge of the average mass of a nebula but solely from the value of R_0 which, in its turn, depends on the assumption that the equation $K = 1{\cdot}83\delta$ is exact. The density turns out to be 1000 times greater than that calculated directly (3.29). This implies that the main bulk of the material content of the universe consists, not of the luminous nebulæ, but of dark, invisible matter. It is interesting to note that, if future observation gave a highly accurate value of K, we should have here a method of determining the average density of matter in space whether the matter were luminous or not. This determination would rest on the number of the luminous nebulæ, on their average red-shift and on the colour of their radiation.

4. *Hyperbolic Space*

We may summarise as follows our analysis of Hubble's method of interpreting the nebular counts :—

(*a*) Five different definitions of "distance" are employed which we have denoted by D, D_1, D_2, D_3, D_4.

(*b*) Of these, D_3 is identified with a radial co-ordinate r of the model universe on the ground that $K = 1{\cdot}83\delta$ exactly. The radius of curvature of space in the model is, however, so small that r is not identical with distance.

(*c*) The method makes the entire structure of the universe turn on the above value of K : a comparatively slight change in this quantity making very great differences in the radius of space and the character of the curvature.

In order to overcome these difficulties,† the present author has proposed an alternative method of comparison of observation and theory which employs, instead of an *a priori* value of K, an *a priori* value of the average mass of a luminous nebula. The method therefore makes no appeal to the concept of distance, but relies on the empirical formulæ (2.1), (2.2) and (2.9), which are in fact established independently of distance considera-

* E. Hubble, *M.N.*, **97**, 506, 1937.
† G. C. McVittie, *loc. cit.*

tions. The metric of the resulting model universe is such that

$$k = -1, \quad R_0 = 1{\cdot}5 \times 10^9 \text{ parsec}, \quad \rho = 8{\cdot}29 \times 10^{-30\pm1} \text{ gr./cm.}^3, \tag{4.1}$$

so that space is hyperbolic. The important features of this result are that it accepts the luminous nebulæ as the main content of the universe, and that it is very insensitive to the value adopted for the average mass of a nebula. Taking mean values of the constants in (3.1) and (3.2), it is easy to show that the hundredfold change of mass from $2 \times 10^9\odot$ to $2 \times 10^{11}\odot$ changes R_0 from 1×10^9 to 2×10^9 parsec and leaves the value -1 of k unaltered.

It is possible to work back from (4.1) to the associated definition of the co-ordinate r in terms of m and consequently to the values of η and K. To do this let us write (3.32) in the form

$$\delta = \tfrac{1}{2}g_0'r - \tfrac{1}{4}g_0''r^2, \tag{4.2}$$

regarding for the moment the quantity $krg_0'/12R_0^2$ as negligible compared with $g_0''/4$. We also assume that the definition of r in terms of m is given by (2.13), viz.

$$\log_{10} r = 0{\cdot}2(m - \eta\delta) + Q.$$

Hence the relationship between δ and r from the observational point of view is (2.14), and therefore equating coefficients of r and r^2 in (4.2) and (2.14), we obtain

$$\tfrac{1}{2}g_0' = 10^{-(X+Q)} \tag{4.3}$$

$$-\tfrac{1}{4}g_0'' = 0{\cdot}2E(\eta - 3)10^{-2(X+Q)}. \tag{4.4}$$

Hence

$$\tfrac{1}{4}g_0'' = 0{\cdot}2E(3 - \eta)(\tfrac{1}{2}g_0')^2. \tag{4.5}$$

But in the course of calculating (4.1) it can be shown * that

$$2\{\tfrac{1}{2}g_0'' + (\tfrac{1}{2}g_0')^2\} = (1{\cdot}55 \pm 0{\cdot}18)(\tfrac{1}{2}g_0')^2.$$

Combining this result with (4.5), it follows that in the hyperbolic universe (4·1)

$$\eta = 3{\cdot}25 \pm 0{\cdot}10. \tag{4.6}$$

If we write $Q = 1 - 0{\cdot}2M_1$, it follows that the definition of r in terms of m is now

$$\log_{10} r = 0\ 2(m - 3{\cdot}25\delta - M_1) + 1, \tag{4.7}$$

δ being given by (3.2). Alternatively, putting $Q = 1 - 0{\cdot}2M_2$, $\delta \equiv \delta_2$, we have

$$\log_{10} r = 0{\cdot}2(m - 3{\cdot}25\delta - M_2) + 1. \tag{4.8}$$

The legitimacy of these definitions naturally depends on the possibility of showing that the quantity $krg_0'/12R_0^2$, neglected in (4.2), is in fact smaller

* G. C. McVittie, *loc. cit.*, formula (3.11). Note that in this formula R^2 is proportional to e^g.

than $g_0''/4$. To verify this, note that (4.3) gives, as in the calculation of (2.15),

$$\tfrac{1}{2}g_0' = 1{\cdot}8 \times 10^{-9}.$$

Also at $m = 21{\cdot}03$, (3.2) gives $\delta = 0{\cdot}230$ and (4.7)

$$\log_{10} r = 8{\cdot}1. \tag{4.9}$$

Hence, by (4.1),

$$\frac{k}{12}\frac{rg_0'}{R_0^2} = -9{\cdot}32 \times 10^{-3}(\tfrac{1}{2}g_0')^2.$$

But by (4.5), (4.6),

$$\tfrac{1}{4}g_0'' = -1{\cdot}15 \times 10^{-1}(\tfrac{1}{2}g_0')^2.$$

Hence at the outer limit of the survey the approximation (4.2) is still valid if the counts be interpreted by means of a hyperbolic universe. Again, turning to the connection between r and distance l, it follows by (4.9) and (4.1) that

$$\frac{k}{6}\frac{r^2}{R_0^2} = -1{\cdot}17 \times 10^{-3}.$$

This quantity is negligible compared with unity, and therefore, by (3.8), $l = r$. We conclude that, in the hyperbolic model (4.1), the definitions (4.7), (4.8) of the co-ordinate r are valid, and, moreover, r is identical with l to a high degree of approximation.

If we now compare (4.7) and (4.8) with (3.6), (2.9) and (2.10), which define D, D_1, D_2 respectively, it appears that r is more nearly identifiable with these quantities than with either D_3 or D_4. We may therefore say that the theory of the hyperbolic model leads to the conclusion that the distances, as determined by rigid scales, of the extra-galactic nebulæ are given by D, D_1 or D_2. We have already pointed out that the empirical formulæ themselves indicate this to be the case.

Lastly, we consider the value of K predicted by the hyperbolic model (4.1). Since $\eta = 3{\cdot}25 \pm 0{\cdot}10$, it follows by (3.12) that *

$$K = (1{\cdot}08 \pm 0{\cdot}10)\delta. \tag{4.10}$$

This is to be compared with Hubble's direct determination giving $K = 1{\cdot}83\delta$. In view of the necessarily approximate character of the calculations on which the latter is based, it is hardly possible to regard (4.10) as lying outside the bounds of possibility.

Summary

The empirical formulæ giving the numbers of extra-galactic nebulæ and the red-shift in terms of apparent photographic magnitude are analysed and shown to imply at least five different definitions of distance. Hubble's small spherical universe depends on the choice of one of these definitions

* Corresponding, according to Hubble, to effective temperatures between 7000° ($K = 1{\cdot}23\delta$) and 7500° ($K = 0{\cdot}83\delta$) (*Ap. J.*, **84**, 517, Table VI, 1936).

dictated by the hypothesis that the equation $K = 1{\cdot}83\delta$, relating the red-shift δ to the conversion-factor K for passing from bolometric to photographic magnitude, is strictly accurate. Moreover, Hubble's method leads to a value for the total density of matter in space, which turns out to be one thousand times greater than that calculated from the luminous nebulæ alone. On the other hand, if no *a priori* definitions of distance are used so that the value of K need not be known, but if the average mass of a luminous nebula is regarded as known, the nebular counts lead to a hyperbolic universe. In this universe it can be shown that the value of K must be $(1{\cdot}08 \pm 0{\cdot}10)\delta$.

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AN ABSORPTION LINE AT λ 4430·6 OF POSSIBLY INTERSTELLAR ORIGIN.

C. S. Beals and G. H. Blanchet.

Introduction.—Until comparatively recently the only lines observed in stellar spectra which were known to be of interstellar origin were the principal doublets λλ 3968·46, 3933·66 (H and K) of *Ca*II and λλ 5895·92, 5889·96 (D_1 and D_2) of *Na*I. As compared with most absorption lines in early type spectra these lines are unusually sharp and well defined, so much so that sharpness has been commonly regarded as an essential quality of interstellar lines. The sharpness of the interstellar lines has been interpreted as a natural consequence of the physical state of interstellar matter, where the density is extremely low and where other influences likely to widen the lines (such as Doppler effects) are expected to be of a lower order of magnitude than those found in the atmospheres of most early type stars. Recent observations by one of the writers * have indicated that some of the lines of interstellar calcium and sodium are wider than had formerly been supposed, but the measured widths are still too small to be comparable with stellar lines occurring in the same spectra.

Recent observations at Mount Wilson have considerably increased the number of known interstellar lines. In 1934 Merrill † announced the discovery of four lines in the visible region, of wave-lengths λλ 5780·4, 5796·9, 6283·9 and 6613·9, which behaved like interstellar lines with regard to their occurrence, their intensities and their stationary character. Instead of being narrow and sharp, however, like the *Na*I and *Ca*II lines, they were somewhat widened and had rather diffuse edges. The diffuse character of these lines naturally gave rise to some question as to their interstellar origin, but in a later paper Merrill ‡ presented evidence which demonstrated their stationary character beyond reasonable doubt. Since a telluric origin can presumably be ruled out, it now seems reasonably certain that they are actually interstellar lines. § So far the chemical identification of the lines has not been made. In the same paper Merrill referred to "a vague feature near λ 4427" which he suspected to be an interstellar line, although no further details concerning it are given.

More recently a number of new interstellar lines have been discovered by Dunham and Adams.‖ The wave-lengths and identifications of these lines are as follows: λλ 3302·28, 3302·98 *Na*I; λλ 7664·94, 7699·01 *K*I;

* *M.N.*, **96**, 661, 1936.

† *P.A.S.P.*, **46**, 206, 1934.

‡ *Ap. J.*, **83**, 126, 1936; *Mt. W. Cont.*, No. 536.

§ In a paper which has just appeared Merrill and Wilson add two new lines, λ 6202·99 and λ 6269·99, and further convincing evidence of interstellar origin (*Ap. J.*, **87**, 9, 1938).

‖ *P.A.S.P.*, **49**, 26, 1937; *Nature*, **139**, 246, 1937.

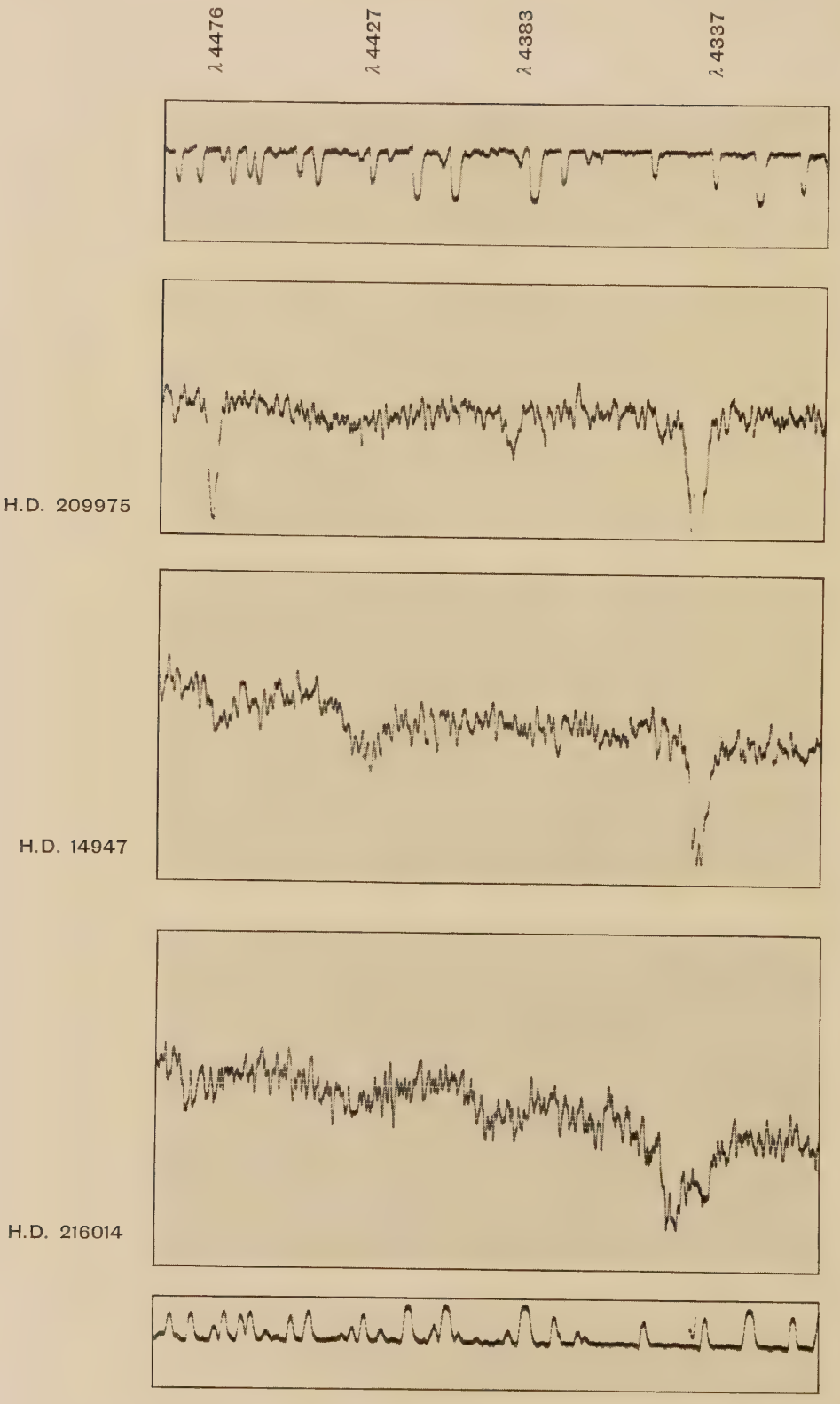


FIG. 1.—*Tracings showing Occurrence of λ 4430 in Stellar Spectra.*

λ 4226·73 *Ca*I ; λλ 3073·97, 3229·18, 3241·99, 3383·78 *Ti*II ; λλ 3957·7, 4232·6, 4300·3 unknown. All of these lines are narrow and sharply defined, and their suggested identification as interstellar lines apparently rests to some extent on their sharp character.

The appearance in stellar spectra of the sharp interstellar lines enumerated above is not entirely unexpected. They are all similar in character to the well-known *Ca*II and *Na*I lines, while those which have been chemically identified are due to terrestrially abundant atoms and represent electronic transitions from the ground-level of the atom. In the appearance of diffuse lines, on the other hand, we are presented with an entirely new problem, particularly since the lines have not been identified and we cannot be certain whether they are due to atoms or molecules. For this reason it seems desirable to secure as many observations as possible, particularly where the hypothesis of interstellar origin may be open to some question. With this object in view the writers have made a detailed study of the line λ 4430·6, the feature in the vicinity of λ 4427 referred to by Merrill.

Observational Data.—The line λ 4430 was first observed at this observatory on a microphotometer tracing of the spectrum of the star H.D. 108 (type O6q, Mag. 7·36), which was under observation because of the P Cygni character of its lines. The stellar lines of this star are rather narrow for an O-type spectrum, and the great breadth of λ 4430 makes it a fairly conspicuous feature on the tracing, even though the central intensity is small. The obvious lack of similarity between this line and other lines in the spectrum was very puzzling, but the idea that it might be of interstellar origin was not seriously considered until the appearance of Merrill's announcement.

Fortunately there was already available at this observatory a large number of spectra of distant O- and B-type stars. Most of these spectra were secured for the investigation of the interstellar K line, and since they include the region from λ 3900 to λ 4900 they are equally well suited for the investigation of λ 4430. It was accordingly decided to confine the investigation to these plates. All of the plates secured since 1929 had calibration spectra imposed on them either with the aid of an optical wedge or a rapidly rotating step sector, and for the most part only these plates have been used. For some of the wave-length measurements older plates of excellent quality but with a less satisfactory calibration have been employed.

The line λ 4430 is so broad and diffuse that it can seldom be detected by an eyepiece examination of the plate, but reveals itself on the tracing as a slight dip in the continuous spectrum of the star.

In fig. 1 (Plate 4) an array of tracings of stellar spectra is shown. On these the line appears with varying intensities. Since the tracings chosen for illustration are among those which show the line most distinctly, it can readily be understood that there are many instances where the actual presence or absence of the line is a matter of uncertainty. Experience in the examination of tracings eventually leads to a certain degree of confidence in distinguishing between real and spurious lines, but many of the measures are necessarily subject to very large probable errors. The profiles are complicated by the presence of stellar lines which are either superposed directly on the

broad profile of λ 4430 or are so placed that they make it difficult to fix the limits of the wings. Those most commonly occurring are the doublet at λ 4416 due to *O*II, the helium line λ 4437 and two narrow lines of wave-length λ 4426·6 and λ 4430·9, which are sometimes blended to form a single line. No identification of these lines has been discovered, but a study of their occurrence in stars of different types revealed that they occur more frequently and with greater intensity in stars of later type, hence it seems probable that they are stellar lines which have hitherto escaped detection. While the possibility has been considered that they might be structural features of the line under investigation, a stellar origin appears more probable and they have been disregarded as far as possible in drawing the profile of the wider line.

Numerical Results.—In fig. 2 two typical profiles of the line in question are shown. These illustrate the great width of the line as well as its

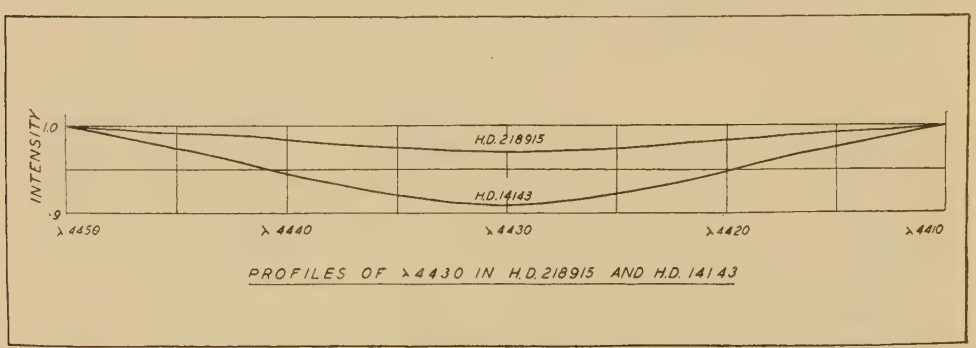


Fig. 2.

variation in depth. The intensities are determined by making planimeter measurements of the plotted profiles. All intensities are given in terms of the equivalent width in km./sec. The relationship 1 A. = 67·7 km./sec. at this wave-length makes it easy to transform them to angstroms if desired. The limit of detection seems to be at about intensity 50 km./sec., and in some cases where the presence of the line was suspected, but where the intensity was too weak to allow of significant measures, an arbitrary value of 40 was assigned for purposes of averaging.

Measures of wave-length have been derived from those tracings on which the line appeared to be reasonably definite and well defined. All tracings included records of the iron comparison lines as well as of the star spectrum, and the measurements were made directly on the tracings by means of a millimeter reseau. In spite of the great width and low central intensity of the line the wave-length measures were reasonably consistent, and individual plates seldom varied from the mean by more than two angstroms. The usual corrections for the Earth's orbital motion were applied, though these corrections seldom amounted to more than a few tenths of an angstrom.

In Table I the results of both intensity and wave-length measurements are given. The headings of the columns in this table are, for the most

part, self-explanatory. Individual weights were assigned to each tracing. These were divided into four classes with weights 1, 2, 3 and 4 respectively. Weight 1 corresponds to a barely detectable line, the form of which is impossible to define with any accuracy. Weight 2 corresponds to a line well

TABLE I

Star H.D.	Spectral Type	No. Plates	Total Wt.	Intensity λ 4430	Probable Error	Wave-length	Intensity *Ca*II 3933
108	O6	6	12	218	8·7	4429·8	38
1337	Bo	4	0	Trace	...	...	36
5005	O6	3	12	77	3·1	4431·0	53
13841	B2	5	18	130	6·4	4429·9	55
13854	Bo	8	32	173	4·0	4431·9	55
14134	Bo	4	16	147	0·9	4430·1	66
14143	Bo	5	16	256	10·5	4430·8	69
14633	O8	1	4	121	...	...	35
14818	B2	6	18	115	4·5	4430·5	62
14947	O5	2	8	152	1·6	...	38
14956	B1	4	16	261	3·8	4430·4	52
17505	O7	1	4	163	...	4431·7	49
23180	B1	4	8	41	2·9	...	8
24912	O7	6	8	75	1·6	...	15
30614	Bo	5	8	93	2·7	4431·0	29
36486	Bo	2	0	0	...	...	4
36861	O8	5	0	40	...	...	17
37043	O8	2	0	0	...	...	10
47129	O8	3	9	132	4·6	4429·7	28
47839	O7	5	0	0	...	...	19
149881	B2	5	8	61	2·9	...	25
166182	B3	3	5	80	5·9	...	18
168957	B3	2	0	Trace	...	...	22
186980	O8	1	3	77	...	...	29
188001	O7	3	3	71	3·3	...	27
188209	O8	4	0	Trace	...	...	28
190429*S*	O9	3	12	109	1·9	4430·8	48
190429*N*	O5	4	16	141	12·3	4427·9	52
190864	O6	3	3	95	7·4	...	56
190918	O9	4	12	145	0·5	4428·3	50
191201	Bo	1	1	117	...	...	36
192163	W.R.	3	9	144	13·4	...	36
192639	O7	1	4	121	...	...	24
193322	O8	2	4	98	...	4430·3	37
193514	O8	1	4	209	...	...	45
193793	W.R.	3	3	59	...	...	50
198846	O9	2	5	112	...	...	23
199579	O6	1	0	Trace	...	...	26
202214	O9	2	3	65	...	4430·3	31
209481	O9	4	10	72	...	...	31
210839	O6	4	12	101	...	...	25
214680	O9	1	1	57	...	...	20
216014	Bo	6	12	181	...	...	37
217297	Bo	1	3	186	...	...	37
218915	O9	4	12	97	...	4430·8	39
35–3955	B1	1	3	53	...	...	39

marked but with poorly defined form and limits. Weight 3 refers to a line which is moderately strong and well defined but with some uncertainty about the limits of the wings. Weight 4 is assigned to the best lines where there is little uncertainty about either the central intensity or the extent of the wings. The total weight, in column 4, represents the sum of the weights of individual tracings. Not all the tracings measured for intensity have been used in the wave-length measures, but the weights of the wave-lengths where they are given are, in general, the same as those of the intensities. The last column of the table contains the intensities of the interstellar K lines taken from a publication * already mentioned. These intensities are expressed in the same units (km./sec.) as those of λ 4430.

Relation between Intensity and Distance.—An effective test of the interstellar character of a line appearing in a stellar spectrum is the relationship between the line intensity and the distance of the star. An hypothesis of interstellar origin requires a definite increase in intensity with distance, while lack of any evidence of such correlation should practically eliminate the possibility that the line is due to absorption in interstellar space. A convenient way to demonstrate the presence or absence of such a correlation is by means of a direct comparison between the intensity of λ 4430 and that of interstellar K, a line whose intensity is known to increase with distance. This has been done graphically in fig. 3. On this graph the ordinates represent K intensities, while the abscissæ are the intensities of λ 4430. Values of distance corresponding to the K intensities are also marked on the graph. These have been derived from a distance-intensity curve prepared by one of the writers.† Each filled circle represents a star, while the diameters of the circles correspond to the total weights as given in Table I. Although the scatter of the points is large there appears to be reasonably definite evidence of a correlation between the two sets of intensities, indicating an increase of the intensity of λ 4430 with increasing distance. The slope of a straight line drawn among the points is such as to indicate that the total intensity of λ 4430 is, on the average, about 3 times that of K. The graph gives some evidence of an increase of λ 4430 with distance somewhat more rapid than that of K, but measures for the distant star are not sufficiently numerous to make this certain. While some of the measures of Table I may represent spurious lines, the general character of the graph of fig. 3 is difficult to explain except in terms of absorption by interstellar material. Particularly convincing are the results for distant and very close stars. Stars known to be distant without exception show a fairly strong absorption line at λ 4430, while relatively close stars, such as the brighter stars in the Orion region, show little or no evidence of the presence of the line. Accordingly, even though the width and form of the line differ widely from those of the more familiar interstellar lines, the hypothesis that λ 4430 is of interstellar origin must receive serious consideration.

Constancy of Wave-length.—The diffuse character of λ 4430 makes difficult the establishment of its stationary or non-stationary character, and

* *M.N.*, **96**, 661, 1936.

† *P. Dom. Ap. O.*, **6**, 336, 1937.

it is unfortunate that in none of the single-lined binaries which are available is the line sufficiently definite to permit tests of the required accuracy.

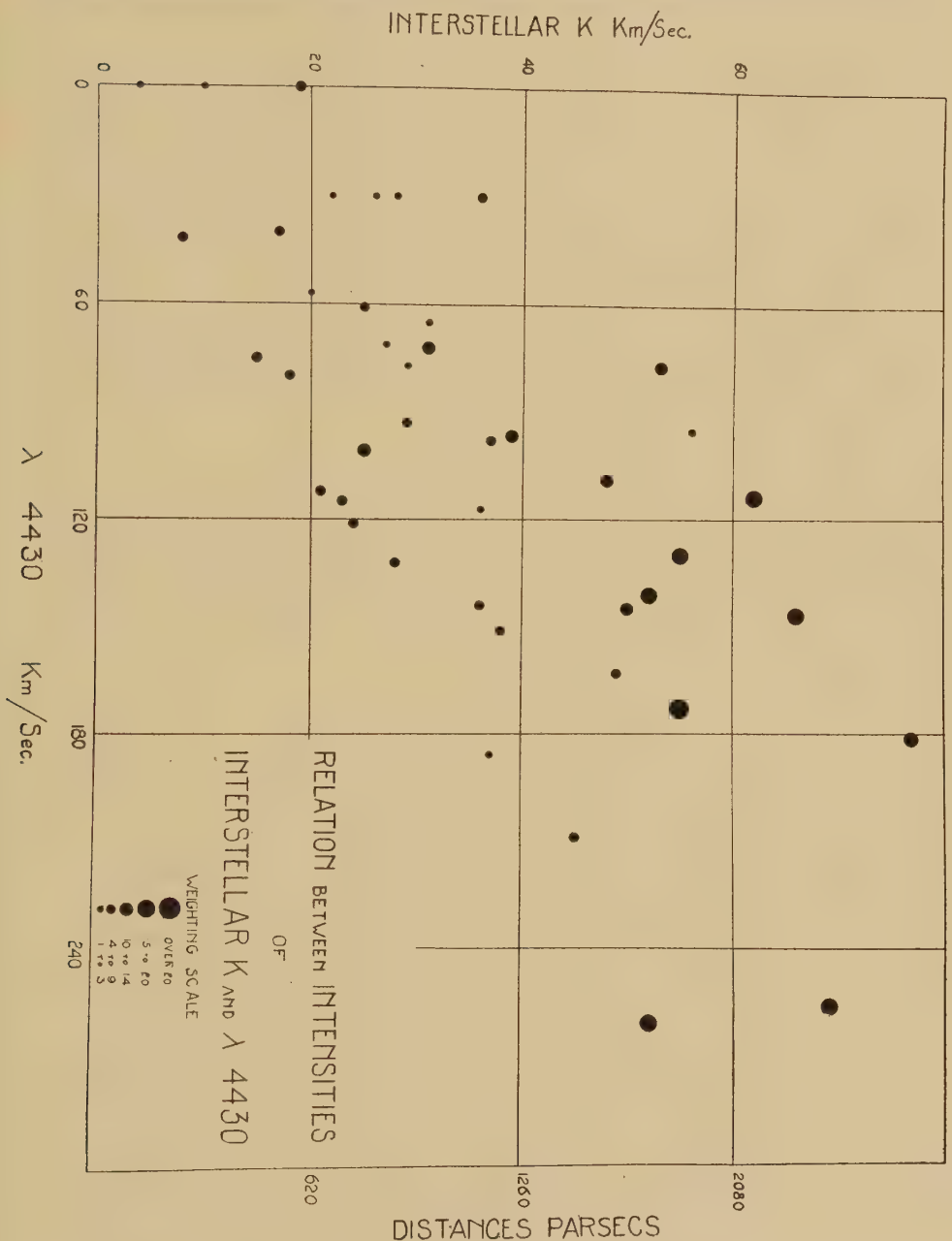


Fig. 3.

There is, however, one double-lined binary, H.D. 216014, in which the line is unusually strong and well defined, and an attempt has been made to use this star to test the stationary character of λ 4430. The orbit of this star

has been determined by Pearce,* who kindly made available to the writers the plates on which the orbit was based. The lines have a separation at maximum of over 400 km./sec., while the relative luminosity of the two components (derived from line-intensity ratios by a method due to Petrie †) is 0·58 : 1. Using this data in conjunction with the known form of λ 4430 it is possible, on the assumption that the line is stellar, to compute the profile of the blend due to primary and secondary components, and to find the variation in mean wave-lengths corresponding to the change from maximum to minimum velocity of the primary star. The observational material is shown in Table II, which contains the velocities of primary and secondary components for two groups of five plates each and the wave-lengths of λ 4430 which were actually observed.

Assuming that the spectra of primary and secondary components are identical and using the profile of λ 4430 derived from spectra taken at the single-lined phase, the profile and velocity of the blended line was computed on the basis of the formula

$$I = R_{1\lambda}L_1 + R_{2\lambda}L_2. \qquad (1)$$

In this formula R_1 and R_2 are intensities at wave-length λ of the lines due to the primary and secondary components respectively. These intensities are expressed as fractions of the continuous spectrum, L_1 for the primary and L_2 for the secondary component. By a suitable choice of units

$$L_1 + L_2 = 1,$$

and since

$$L_2 = 0{\cdot}58L_1$$
$$L_1 = 0{\cdot}633 \quad \text{and} \quad L_2 = 0{\cdot}367.$$

TABLE II

Group	Plate Number	Primary Velocity	Secondary Velocity	Observed Wave-length
	11091	+192	−273	4429·2
	10956	+103	−185	4427·3
1	11063	+172	−239	4429·8
	11012	+207	−274	4431·3
	10934	+204	−290	4429·1
	Means	+176	−252	4429·3 ± 0·43
	10891	−228	−239	4426·3
	10917	−158	+145	4429·2
2	10983	−236	+236	4429·2
	11083	−235	+216	4429·2
	11143	−228	+214	4431·4
	Means	−217	+210	4429·1 ± 0·81

* *P. Dom. Ap. O.*, **3**, 171, 1925.
† *P. Am. A.S.*, **8**, 220, 1935.

The observations of Table II were made when the lines of the binary were near maximum separation. The plates are divided into two groups. Group 1 corresponds to primary positive and group 2 corresponds to primary negative. The mean velocities for groups 1 and 2 are used to fix the positions of the lines due to primary and secondary components, and equation (1) is used in conjunction with the known form of λ 4430 to derive the position of the hypothetical blended line on the assumption of stellar origin. The result leads to a positive velocity of +2 km./sec. for group 1 and a negative velocity of −50 km./sec. for group 2, corresponding to a wave-length difference of 0·77 A. The wave-length difference actually observed is only 0·2 A., but while this result is consistent with a non-stellar origin for the line, the size of the probable errors ± 0·4 and ± 0·8 A. is too great to enable any definite conclusions to be drawn.

A second test was made by selecting two groups of eight stars each, the mean velocities of which differed by approximately 38 km./sec., as shown in Table III. The difference in wave-length of λ 4430 for the two groups computed on the basis of a stellar origin is 0·55 A. The wave-length difference actually observed is only 0·03 A., but again the size of the probable errors renders the result inconclusive. These tests indicate slight possibility of establishing a stellar or non-stellar origin for λ 4430 on the basis of wave-length measures, and the tentative conclusion that the line is due to interstellar absorption rests for the present entirely on the correlation between the intensities of λ 4430 and those of the K line, as indicated in fig. 3.

TABLE III

H.D. Star	No. Plates	Stellar Velocity	*Ca*II Interstellar Velocity	Observed Wave-length
108	6	−64	−20	4429·8
13841	5	−39	−28	4429·9
13854	8	−40	−26	4431·9
14134	4	−43	−31	4430·1
14143	5	−45	−26	4430·8
14818	6	−47	−24	4430·5
14956	4	−30	−21	4430·4
218915	4	−72	−11	4430·8
	Means	−47·5	−23·4	4430·52 ± ·18
5005	3	−21	−17	4431·0
17505	1	−17	−19	4431·7
30614	5	+ 6	− 6	4431·0
190429 *N*	3	− 6	−11	4427·9
190429 *S*	4	−17	−14	4430·8
190918	4	+ 3	−17	4428·3
193322	2	− 6	−13	4432·7
202214	2	−16	−19	4430·3
	Means	− 9·2	−14·5	4430·55 ± ·38

Chemical Identification.—As yet no definite chemical identification has been found for λ 4430. Its great width makes it difficult to associate the line with atomic absorption. The only agency capable of producing a wide atomic line under interstellar conditions would appear to be radial motion of absorbing atoms, and atomic lines which are known to be interstellar show no evidence of the existence of velocities of the required magnitude. The real width of the widest atomic interstellar lines, apart from that produced by instrumental distortion, is probably of the order of 100 to 200 km./sec. λ 4430 is more than ten times this width, and there seems little possibility of reconciling this with an atomic origin.

Other alternatives are absorption by molecules of gas or by particles of solid material. Possibilities for molecular absorption have been discussed by Russell,* Eddington,† Swings,‡ Saha,§ Swings and Rosenfeld ‖ and Eyster.¶

There appears to be general agreement that molecules due to terrestrially abundant atoms can exist in interstellar space, and a number of coincidences between molecular wave-lengths and interstellar absorption lines have been pointed out. While none of the suggested identifications can yet be regarded as certain, this appears to be a promising approach to the problem, and some at least of the newly discovered interstellar lines may be due to common molecules. If this should prove to be the case, then the probability that λ 4430 is also of molecular origin would be considerably increased. However, a molecular origin can only be regarded as certain when the wave-length and profile of the line can be accurately correlated with the detailed structure of a band either observed in the laboratory or calculated from molecular constants derived from laboratory data.

Owing to the variety of structures encountered in molecular spectra the profile of λ 4430 does not give any very definite clue to its identification. It may be noted, however, that the symmetry of the line does not correspond to the form commonly associated with electronic bands of diatomic molecules, which is definitely unsymmetrical. From this it may be inferred that the band is probably not due to an electronic band of a diatomic molecule. There remain as possible origins the vibration bands of diatomic molecules and band systems due to polyatomic molecules, but these exhibit such a diversity of forms that neither could be excluded as a possible carrier of the band.

Although it was the great width of λ 4430 which first suggested a molecular origin, recent theoretical discussions lead to the paradoxical situation that its width may actually be an embarrassment in an attempt to interpret the line in terms of absorption by molecules of interstellar gas. Swings has suggested that under the temperature conditions of interstellar space only

* *M.N.*, **95**, 635, 1935.
† *Observatory*, **60**, 99, 1937.
‡ *M.N.*, **97**, 212, 1937.
§ *Nature*, **139**, 840, 1937.
‖ *Ap. J.*, **86**, 483, 1937.
¶ *Ibid.*, **86**, 486, 1937.

a few of the lower rotational states of the molecule will be occupied. If this is so, only a few of the band lines would be expected to appear in absorption, and the resulting band should not be of great width. While general theoretical considerations as well as observations of atomic interstellar lines (notably the lines of *Ti*II) are in agreement with a low temperature distribution of molecules in rotational states, the difficulty should perhaps not be taken too seriously until it has a more definite observational basis. Until a few unquestioned identifications have been made the whole matter of molecules in interstellar space must remain in a state of extreme uncertainty.

In the absence of a definite identification in terms of absorption by gaseous molecules, the possibility that λ 4430 may be due to absorption by particles of solid material should not be entirely disregarded. It is well known that many transparent solids and liquids exhibit the property of selective absorption, as this property is often utilized for making colour-filters. In a majority of instances the absorption bands are of great width, of the order of hundreds or thousands of angstroms; but some substances, notably the salts of rare Earth elements,* have absorption bands which are relatively narrow, some even having widths of the same order as atomic lines in stellar or laboratory spectra. Since the obscuration observed in various regions of the sky is commonly attributed to particles of solid material, it is within the realm of possibility that selective absorption by such particles may result in the appearance in stellar spectra of observable lines as bands. In so far as can be determined from the literature available to the writers, the optical properties of many of the lighter elements in their solid form are insufficiently determined, and the same is true of many compounds. Until more complete data of this sort are available, suggestions relating interstellar absorption lines to solid particles can hardly advance beyond the stage of speculation.

Summary.—A broad ill-defined line about 40 A. in width and of approximate wave-length λ 4430·5 has been observed in the spectra of a number of early type stars. The equivalent width of the line is shown to exhibit a marked correlation with that of interstellar K, suggesting that λ 4430 may also be of interstellar origin. The diffuse character of the line has so far rendered unsuccessful any attempts to establish its stationary or non-stationary character in spectroscopic binaries. Possibilities for the origin of the line would appear to be limited to absorption by gaseous molecules or by particles of solid material in interstellar space.

Dominion Astrophysical Observatory,
Victoria, B.C.:
1938 *February* 8.

* Spedding and Bear, *Phys. Rev.*, **46**, 308 and 975, 1934.

ON THE ACCURACY OF PARALLAXES DETERMINED FROM SPECTRAL CLASS AND PROPER MOTION.

W. J. Luyten.

Parallaxes estimated from proper motion and spectral class have been widely used in statistical investigations, as it has long been known that these will give accurate determinations of the mean distances of groups of stars. It does not appear to have been recognized that these estimates possess a high individual accuracy which, even for moderately large parallaxes, compares favourably with the accuracy of the trigonometric determinations.

TABLE I

Star No. (*L.O.B.*, 344)	Parallax		Star No. (*L.O.B.*, 344)	Parallax	
	Hyp.	Trig.		Hyp.	Trig.
	″	″		″	″
10	0·042	0·083	299	0·048	0·060
18	105	103	317	33	32
22	17	87	333	34	50
*69	43	38	337	73	51
75	91	73	346	125	50
113	44	24	410	(10)	16
134	66	64	411	36	17
141	40	33	412	105	79
150	40	36	415	87	61
153	33	18	433	69	50
179	29	30	440	(25)	−58
183	33	27	456	(10)	19
189	(14)	63	463	46	41
191	76	161	466	115	150
200	55	48	501	105	255
203	30	10	515	83	86
210	38	34	551	87	102
229	46	38	570	52	78
230	29	28	576	50	57
239	40	28	587	33	12
251	55	37	590*a*	50	41
258	30	65	592	(10)	7
268	125	65	610	115	76
274	32	−5	630	42	38
278	40	48	637	52	10:
284	50	9	707	96	207

* The determination in *L.O.B.*, 351 (1923), of the spectrum of this star seems to have been unknown to Reuyl, who in *A.J.*, 1057, draws attention to the large space velocity and expresses the hope that its spectral class may be determined soon. Reuyl's space velocity is based on the McCormick trigonometric parallax which verifies the spectral parallax determined in 1922.

To test this statement it is no use, of course, to take a large number of stars of known parallax, spectrum and proper motion, and determine from them the reduction curves for mean parallax—empirically or analytically—and then to compare the parallaxes thus derived with the formula with those originally observed. The only real test of the method lies in the prediction of parallaxes purely from spectral class and proper motion, on the basis of reduction curves determined from other stars, and the subsequent verification of such predictions by the trigonometric method.

Fortunately two such series of test cases are available. In 1923 I published the spectra and hypothetical parallaxes for 100 stars of large proper motion * and in 1924 a list of hypothetical parallaxes for bright stars in the southern hemisphere.† At the time these lists were published only a small proportion of the stars had known trigonometric parallaxes, but since that time trigonometric parallaxes have been determined for all but a few of the remainder.

Table I shows the comparison between the parallaxes predicted by me (second column) and those subsequently determined by the trigonometric method (third column, absolute values), while the stars are identified by the same number as in *L.O.B.*, 351, *i.e.* by the numbers assigned to them in my proper motion catalogue in *L.O.B.*, 344. For five stars in the list the spectral-proper motion parallaxes are bracketed as they were in *L.O.B.*, 351, since I then felt that these stars were too exceptional to expect the method to work for them. One of these has since turned out to be a white dwarf, two are white "intermediates," and the other two F–G stars of very large tangential velocity.

In a similar manner Table II (p. 410) gives the comparison between predicted and observed parallaxes for 77 stars taken from *H.C.*, 251.

A casual glance at the tables shows that the agreement is good, about as good as for the mutual agreement between two different series of trigonometric parallaxes. There are, of course, some black sheep in the list, such as Nos. 22, 346 and 501 in the first list, and p Eridani, α Phœnicis and π Mensæ in the second. For the first of these it is highly probable that either the spectral class used is wrong, or that the star is a yellow infra-dwarf, while p Eridani is a very exceptional star all round with a tangential velocity of only 8 km./sec., and a much larger mass than the mean for its spectral class. And as against these failures of the method one may point to the two negative trigonometric parallaxes for Nos. 274 and 440 in the first list, as evidence that these parallaxes, too, are subject to errors.

If for the first list we reject the five stars whose predicted parallaxes were originally bracketed as well as Nos. 274, 284, 440 and 637, because for them the trigonometric parallax is either negative or very uncertain, but no other stars (retaining even No. 22), we find for the mean error in the difference in absolute magnitude, *i.e.* of $M_{\text{trig}}-M_{\text{hyp}}$ a value of $\sigma = \pm 1^{\text{m}}.1$. Estimating the contribution to this caused by the uncertainties in the trigonometric parallaxes at (conservatively) $\pm 0^{\text{m}}.6$ we find for the mean error of a single

* *L.O.B.*, 351, 1923.
† *H.C.*, 251, 1924.

determination of the parallax by the spectral-proper motion method $\sigma = \pm 0^m.9$.

For Table II we find, similarly, $\sigma = \pm 0^m.97$, and, estimating the part due to the trigonometric parallax errors again at $\pm 0^m.6$, we derive $\sigma_{method} = \pm 0^m.76$.

TABLE II

Star	Parallax		Star	Parallax		Star	Parallax	
	H.C., 251	Trig.		*H.C.*, 251	Trig.		*H.C.*, 251	Trig.
	″	″		″	″		″	″
37 Phe	0.060	0.045	B 2312	0.063	0.062	η Sco	0.058	0.066
α Phe	72	21	b_2 Car	40	51	ξ Oph	54	63
60 Tuc	72	54	ψ Vel	58	70	d Oph	44	17
ν Phe	87	79	175 Hya	76	78	λ Ara	44	40
κ Tuc	63	45	49 Ant	43	46	γ Sgr	40	24
p Eri	40	160	54 Ant	40	50	ι Pav	46	49
χ Eri	105	55	215 Vel	63	31	ϵ Sgr	50	20
22 For	58	64	χ_1 Hya	40	36	λ Sgr	42	35
λ_2 For	40	50	31 Cen	44	42	τ Sgr	44	41
ι Hor	46	74	289 Hya	91	91	242 Sgr	52	71
ϵ For	59	34	66 Cen	144	93	262 Sgr	42	48
κ Ret	63	59	300 Hya	44	15	δ Pav	180	174
24 Ret	46	32	185 Cen	58	45	279 Sgr	280	177
γ Dor	55	69	188 Cen	44	17	86 Tel	42	42
ζ Pic	44	19	191 Cen	58	49	ϕ_2 Pav	66	45
π Men	105	42	ι Cen	63	49	ψ Cap	55	93
72 CMa	47	47	i Cen	69	47	α Oct	48	22
38 Pup	44	47	θ Cen	115	60	119 Pav	47	44
90 CMa	58	40	α Cir	60	54	γ Pav	96	112
60 Pup	40	48	56 Lib	48	30	τ PsA	60	47
159 Pup	69	51	ν_2 Lup	91	63	37 Gru	79	80
212 Pup	42	70	ν_1 Lup	50	27	5 Tuc	50	54
69 Car	55	54	g Lup	52	58	υ Aqr	42	42
B Car	52	52	β TrA	79	82	88 Gru	46	36
α Cha	44	49	ζ TrA	46	86	27 Phe	44	25
11 Pyx	91	81	ϵ Sco	105	53			

These two values still include the dispersion due to errors in the magnitudes and spectra used, and these may be considerable for the first list. I believe, therefore, that it is not too much to say that, if accurate spectral classes, magnitudes and proper motions are available, the parallax of a star may be predicted with a mean error of 23 per cent., corresponding to $0^m.7$ in the resulting absolute magnitudes.

Summary.—Parallaxes computed by the writer many years ago from spectral classes and proper motions are compared with subsequent trigonometric determinations, and it is found that even for an individual star a parallax calculated in this way is as reliable as a single trigonometric determination when the parallax is of the order of $0''.035$.

FURTHER NOTE ON THE SOUTH EQUATORIAL CURRENT OF JUPITER.

A. Stanley Williams.

In a previous note on this current, published in *M.N.*, **96**, 548, 1936, it was shown that there had been a continual decrease in its velocity between the years 1880 and 1910, and that this decrease had probably continued to the year 1926, but with less certainty on account of the insufficiencies of the observational material, due according to Phillips to the absence or scarcity of conspicuous spots on or about the South Equatorial Belt of the planet, and the generally quiescent state of this region of the Equatorial Current. We now have available, however, the observations for two additional years in the recently published *Memoir* of the Jupiter Section of the B.A.A.* These are of very great importance, since they now absolutely prove that the decrease in velocity has in fact been a practically continuous one extending from the year 1880 to the year 1931, a period of no less than fifty-one years.

The additional observations are given below in the same form as before, so as to continue the table on p. 550 of the note previously cited, with the addition of a column containing the weights attributed to the observations. In the original table the observations were all of weight 3, so that this column had been omitted.

Observed Velocities of South Equatorial Current (continued)

Date	Rotation Period †	Velocity km./hr.	Wt.
	h m s		
1929·9	9 50 53·8	−29	3
1931·0	9 50 56·6	−32	2

With regard to these weights there cannot be any hesitation in giving to the first result a weight 3, since it depends upon numerous observations of 14 spots, mostly well observed and distributed nearly uniformly all round the planet. But the other is not so good, and it is only with some reluctance that a weight of 2 has been ascribed to it. But it is in satisfactory agreement with the other.‡

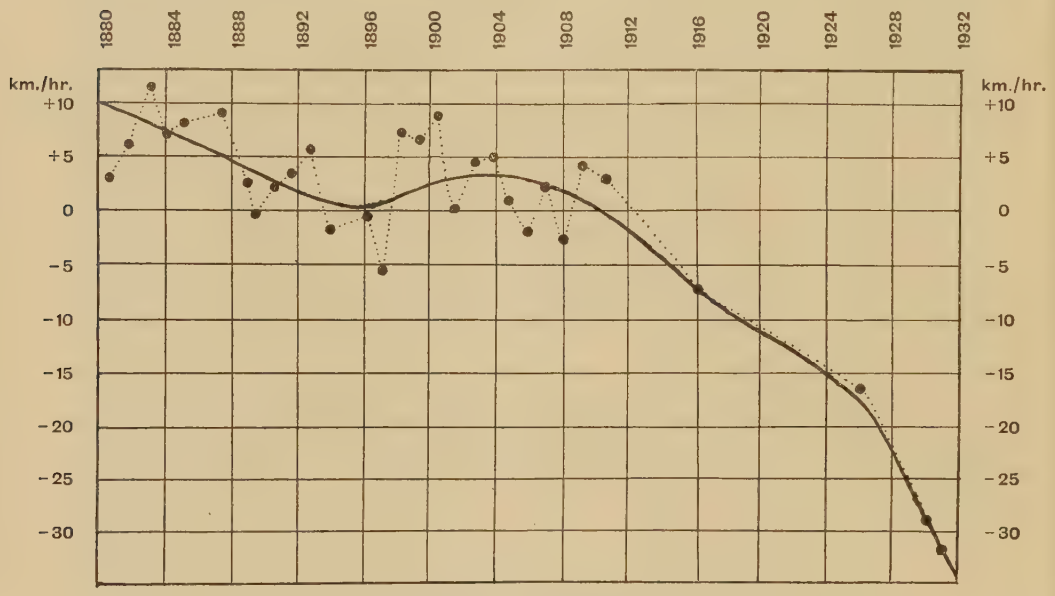
If the two additional observations are added in their appropriate places

* *M.B.A.A.*, **32**, 2, 1937.

† The figures differ slightly from those given by Phillips in the *Memoir*, probably because the periods of the individual spots had been weighted by me according to the intervals of time covered by the observations and the number of such observations, etc.

‡ Observations with a weight 3 are not likely to be in error by more than ± 0·8 km./hr.

to the diagram on p. 551 of the previous note, it will be seen how well they carry on the decreasing velocity of the current, so that in fact such decrease, before a little doubtful after 1910, is now absolutely certain right up to 1931. Between 1880 and 1931, a period of over fifty years, the decrease in velocity amounts to no less than 40 km./hr. The remarkable nature of this decrease is apparent even more clearly in a smoothed or mean velocity curve drawn through the observations. This mean curve is shown by a continuous line in the adjoining diagram; and, with the exception of a pause or slight increase in velocity about the year 1900, it may be said that there has been



Mean Velocity Curve (continuous line) of South Equatorial Current, 1880 to 1931. Zero Velocity corresponds to Rotation Period $9^h\ 50^m\ 30^s$.

a continual decrease in velocity during the fifty years covered by the observations. Nor does the mean curve afford any indication of any periodicity. The remarkable nature of the pause and slight increase in velocity about 1900 seems to merit attention.

The South Equatorial Current is not an insignificant current. Judged by terrestrial standards it is enormous. Roughly it may be said to be 270,000 miles long, with a breadth of 4500 miles. Its depth is unknown. This may be 5 miles, or 10 miles, or even 100 miles or more. We shall probably be on the safe side if we assume the depth to be only 10 miles. Similarly its density is unknown, but having regard to the rapid increase in density with depth in the case of such a planet as Jupiter,* we shall probably be on the safe side again if we assume an average density equal to that of our own atmosphere. We are then faced with the problem of a decrease in the velocity of this enormous current amounting to 40 km./hr.

* In this connection reference should be made to Peek's instructive paper in *M.N.*, **97**, 577, 1937, and to See's paper in *A.N.*, **167**, 114, 1905.

Whatever views may be held as to the probable constitution of the current, or its density or depth, or of the planet's atmosphere in general, it is clear that an enormous change or transfer of energy is here indicated, a change, moreover, apparently quite independent of any solar action or influence, and hence probably having its origin in the planet itself.

Summary.—1. Recent additional observations now show with certainty that the general decrease in the velocity of the South Equatorial Current previously indicated did actually continue to the year 1931, that is for over fifty years.*

2. Between 1880 and 1931 this decrease amounted to 40 km./hr.

3. The mean velocity curve fails to indicate any connection with the solar influences, such as the 11-year sunspot period or the period of the planet's revolution round the Sun,† nor indeed with any long-period periodicity. The change would therefore seem to have originated in the planet itself.

Feock, Truro, Cornwall :
1938 *February* 11.

* It is interesting that a similar secular decrease of velocity, though of much smaller amount, is shown by the observations of the S. Temperate and Southern Currents (*M.N.*, **94**, 675, 1934).

† This is in apparent contradiction to the conclusion previously come to as to the colour changes of the Equatorial Belts synchronising with the period of the planet's revolution round the Sun (*M.N.*, **96**, 552, 1936), but the observed phenomena are very dissimilar, and the colour changes may not involve any great transfer of energy, and hence may be due to some seasonal cause originating in the Sun, or due to its influence.

ON THE FORM OF ROTATING-GAS CONFIGURATIONS.

Zdeněk Kopal, D.Sc.

1. *Introduction.*—The problem of slowly rotating-gas configurations with initially polytropic density distribution has already been extensively dealt with by S. Chandrasekhar.* Their resulting oblateness has been related with the angular velocity of rotation, or, more exactly, with v where

$$v = \frac{\omega^2}{2\pi G\rho},$$

for different degrees of density condensation.† The application of these results to eclipsing binaries—for which alone all the necessary data are known from observations—has led to a rather surprising result, that the density condensations of all discussed stars came out generally too low.‡ But one suspicion suggests itself. In applying results following from the study of polytropes we are open to objection that real stars might be of a constitution very different from that specified by Emden's equation. The very important question arises, now, about the effect of the complete march of density distribution on the resulting oblateness, *i.e.* whether the oblateness (for a given v) is a function of the density condensation alone, or whether it varies when we change the whole march of density distribution, but keep the density condensation constant.

An attempt at solving this problem generally, *i.e.* for any arbitrary density distribution, meets with serious difficulties and would lead to a highly complicated mathematical procedure. To avoid this, we shall limit ourselves to the discussion of three different models which will to some extent cover the ground : the one will be the well-known polytropic model, and others will be defined by special assumptions explained in turn. The reasons which have led to their definitions will be mainly of mathematical character. Apart from their underlying physical significance, these models will be defined so as to present a case which can be dealt with in not too complicated terms.

As will be seen, the method of attacking our problem will be more general and essentially different from that of Chandrasekhar, the starting-point in the present paper being an early investigation of Clairaut. The agreement of the present results in the case of the polytropic model with those of Chandrasekhar leaves no room for doubt that both are generally correct.

* *M.N.*, **93**, 390, 449, 462, 539, 1933.

† See Kopal, *M.N.*, **96**, 854, 1936 ; **97**, 646, 1937.

‡ Under this term we shall understand the ratio of the central to the mean density (or its reversal).

2. *The Equations of the Problem.*—In a series of important papers* Clairaut more than a century ago has shown that for slow rotations the oblateness e of a gas spheroid is (to the first order in e) proportional to the ratio of centrifugal force to gravity at the equator. This is equivalent to

$$e = C \cdot v, \tag{1}$$

where

$$C = \frac{15}{4} \cdot \frac{\phi \int_0^a \rho a^2 da}{\left(a\frac{d\phi}{da} + 2\phi\right)\int_0^a \rho a^2 da - a^3 \int_a^{a_1} \rho d\phi}, \tag{2}$$

which at the boundary for $a = a_1$ clearly becomes

$$C = \frac{15}{4} \cdot \frac{1}{2 + \frac{a_1}{\phi_1}\left(\frac{d\phi}{da}\right)_{a=a_1}}. \tag{2.1}$$

a is the major semi-axis of the spheroid, and ϕ satisfies the following differential equation :—

$$\left(a^2\frac{d^2\phi}{da^2} - 6\phi\right)\int_0^a \rho a^2 da + 2\left(a\frac{d\phi}{da} + \phi\right)\rho a^3 = 0, \tag{3}$$

where ρ is the density, supposed to be a function of a, which vanishes for $a = a_1$, *i.e.* at the boundary of the configuration. Thus the whole problem reduces in fact to the solution of (3).

Several ways for solving it can be suggested. We employ here an elegant treatment due in principle to Legendre and Airy. Introduce a new variable

$$\psi = \phi \int_0^a \rho a^2 da. \tag{4}$$

Hence ψ is a solution of the following equation :—

$$\frac{d^2\psi}{da^2} = \left(\frac{6}{a^2} + \frac{a^2\frac{d\rho}{da}}{\int_0^a \rho a^2 da}\right)\psi. \tag{5}$$

The integration giving ρ as an arbitrary function of a cannot be effected in a general form owing to mathematical difficulties, and for further progress a definite assumption on the march of the density distribution is inevitable.

3. *The Polytropic Model.*—Let us put

$$a^2\frac{d\rho}{da} = \chi \cdot \int_0^a \rho a^2 da, \tag{6}$$

* See Tisserand, *Traité de la Mécanique céleste*, **2**, 211–218, Paris, 1891 ; or Clairaut's original papers.

and determine χ so as to make (6) the equation of a polytrope. Let us start from the hydrodynamical equation :

$$\frac{dP}{da} = -g\rho, \tag{7}$$

where

$$g = G\frac{M_a}{a^2} = \frac{4\pi G}{a^2}\int_0^a \rho a^2 da. \tag{8}$$

Introducing the polytropic relation

$$P = K \cdot \rho^{1+\frac{1}{n}}, \tag{9}$$

and differentiating, (7) becomes

$$a^2\frac{d\rho}{da} = -\frac{4\pi G n}{K(n+1)\rho^{\frac{1}{n}-1}}\int_0^a \rho a^2 da. \tag{10}$$

Introducing further the variables ξ, θ, defined, as usual, by

$$\rho = \rho_c \theta^n,$$
$$a = \left[\frac{K(n+1)}{4\pi G \rho_c^{\,1-\frac{1}{n}}}\right]^{\frac{1}{2}}\xi,$$

(10) reduces to

$$\xi^2\frac{d\theta}{d\xi} + \int_0^\xi \theta^n \xi^2 d\xi = 0. \tag{11}$$

Making now the same substitution in (6), we clearly see that (6) becomes polytropic, if

$$\chi = -\xi_1^{\,2} n \theta^{n-1}, \tag{12}$$

ξ_1 being a constant to be determined later. Thus for the polytropic model eq. (5) takes the form : *

$$\frac{d^2\psi}{da^2} = \left(\frac{6}{a^2} - \xi_1^{\,2} n \theta^{n-1}\right)\psi. \tag{13}$$

There are three cases where (13) can be integrated as it stands, namely, $n = 0$, 1 and 5.

4. *Solution for* $n = 0$ (Maclaurin's model).—If $n = 0$, (13) reduces to

$$a^2\frac{d^2\psi}{da^2} = 6\psi, \tag{14}$$

and has an integral

$$\psi = 6ca^3, \tag{15}$$

c being an integration constant. Then $\rho = \text{const}$,

$$\int_0^a \rho a^2 da = \tfrac{1}{3}\rho a^3; \tag{16}$$

* It is interesting to compare this equation with eq. (37) of Chandrasekhar's first paper (*M.N.*, **93**, 395, 1933).

(4) yields

$$\phi = \frac{18c}{\rho} = \text{a constant}, \tag{17}$$

and therefore

$$\frac{1}{\phi}\frac{d\phi}{da} = 0. \tag{18}$$

Hence from (2)

$$C = \tfrac{15}{8}, \tag{19}$$

in agreement with all earlier investigations.

5. *Solution for* $n = 1$ (Laplace's model).—For $n = 1$, χ reduces to a constant—this case being already discussed from a different point of view by Legendre and also by Laplace * in one of his investigations concerning the internal constitution of the Earth. Indeed it was in this connection that Laplace found the solution of the polytropic equation for $n = 1$.

Eq. (13) takes the form :

$$\frac{d^2\psi}{da^2} = \left(\frac{6}{a^2} - \xi_1^2\right)\psi. \tag{20}$$

Considering the boundary conditions, its solution can be put in the following form :— †

$$\psi = \lambda\left[\left(1 - \frac{3}{\xi_1^2 a^2}\right)\sin \xi_1 a + \frac{3}{\xi_1 a}\cos \xi_1 a\right]. \tag{21}$$

Owing to the density distribution which is now given by

$$\rho = \rho_0 \frac{\sin \xi_1 a}{\xi_1 a}, \tag{22}$$

it follows that

$$\int_0^a \rho a^2 da = \frac{\rho_0}{\xi_1^2}(\sin \xi_1 a - \xi_1 a \cos \xi_1 a). \tag{23}$$

The integration constant λ in (21) is not arbitrary. Remembering that ϕ shall be unity at $a = 0$, we see that

$$\lambda = -\frac{5\rho_0}{\xi_1^2}, \tag{24}$$

and after some minor rearrangements we get

$$\phi = 5\frac{(3 - \xi_1^2 a^2)\sin \xi_1 a - 3\xi_1 a \cos \xi_1 a}{\xi_1^2 a^2 (\sin \xi_1 a - \xi_1 a \cos \xi_1 a)}. \tag{25}$$

So far the boundary value of a remains arbitrary.‡ To lessen this indeterminateness let us put $a_1 = 1$; a glance at (22) will show that in this

* *Cf.* Tisserand, *loc. cit.*, pp. 232–237 and 241–242.

† *Cf.* also Forsyth, *Treatise on Differential Equations*, pp. 197–206.

‡ We are obviously free to choose the boundary value of a arbitrarily ; we have then only properly to adjust ξ_1 so as to make the density vanish at the boundary.

case $\xi_1 = \pi$ (*i.e.* the first zero of the Emden equation for $n = 1$), and (25) consequently yields far-reaching simplifications. The result is that

$$\frac{1}{\phi_1}\left(\frac{d\phi}{da}\right)_1 = \frac{\pi^2}{3} - 2. \tag{26}$$

Hence

$$C = \frac{45}{4\pi^2} = 1{\cdot}1399 \ldots, \tag{27}$$

agreeing exactly with the value arrived at by Chandrasekhar * by a quite different and much more complicated method.

6. *Solution for* $n = 5$ (Roche's model).—The solution of the Emden equation for $n = 5$, according to Schuster, is

$$\theta = (1 + \xi^2)^{-\frac{1}{2}}, \tag{28}$$

and consequently

$$\chi = -5\xi_1^2(1 + \xi^2)^{-2}. \tag{29}$$

As is easily verified,

$$\lim_{n \to 5} \chi = \lim_{\xi \to \infty} 5\xi_1^2(1 + \xi^2)^{-2} = 0, \tag{30}$$

and thus (14) continues to hold good also in this case. But

$$\int_0^a \rho a^2 da = F(\rho), \tag{31}$$

$F(\rho)$ being *ex definitione* independent of a. Hence

$$\phi = \frac{6ca^3}{F(\rho)}, \tag{32}$$

$$\frac{d\phi}{da} = \frac{18ca^2}{F(\rho)}, \tag{33}$$

and, at the boundary,

$$\frac{1}{\phi_1}\left(\frac{d\phi}{da}\right)_1 = 3. \tag{34}$$

Hence

$$C = \tfrac{3}{4}, \tag{35}$$

again in agreement with previous results.

For polytropes other than those specified by $n = 0$, 1 or 5, the results cannot be given in a finite number of terms and the method of solution by means of infinite series must be adopted, setting †

$$\theta^{n-1} = 1 - \frac{(n-1)}{6}\xi^2 + \frac{(n-1)(4n-5)}{180}\xi^4 - \ldots \tag{36}$$

---

* See *M.N.*, **93**, 553, 1933, Table I, col. 6.

† *Cf.* Introduction (by D. H. Sadler) to the *Mathematical Tables, II* (issued by the British Association for the Advancement of Science), on Emden functions.

It is not necessary to enter here into detailed calculations, as this arduous piece of work has already been carried out by Chandrasekhar, and there is no need to repeat it when it is clear from the preceding that the results would agree.

7. *Second Model.*—Let us turn now to the second model and put

$$a^2\frac{d\rho}{da} = -k^2a^m\int_0^a \rho a^2 da, \tag{37}$$

the negative sign on the right side arising from the supposition that the density gradient is negative over the whole range of a. The differentiation of (37) will yield an equation governing the density distribution which satisfies (37):

$$a\frac{d^2\rho}{da^2} + (2-m)\frac{d\rho}{da} + k^2\rho a^{m+1} = 0. \tag{38}$$

For $m=0$, this equation reduces to Emden's equation for $n=1$ and can be satisfied by (22); but apart from this case, the solution of (38) cannot be given in finite terms, being

$$\rho = \rho_c\left\{1 + \sum_{i=1}^{\infty}\frac{(-k^2)^i a^{i(m+2)}}{[3(m+2)][2(m+2)(m+5)]\ \ldots\ [i(m+2)(im-m+2i+1)]}\right\}. \tag{39}$$

The k's in (39) are not arbitrary. Imposing the condition that the boundary is again at $a=1$, they are determined for every value of m. The density condensations of such configurations clearly are:

$$\frac{\rho_m}{\rho_0} = -\frac{3}{k^2}\left(\frac{d\rho}{da}\right)_1. \tag{40}$$

The equation specifying ψ takes now the form:

$$\frac{d^2\psi}{da^2} = \left(\frac{6}{a^2} - k^2a^m\right)\psi, \tag{41}$$

and to solve it, the method of integration by means of infinite series must again be adopted; it yields:

$$\psi = c\,.\,a^3\left\{1 + \sum_{i=1}^{\infty}\frac{(-k^2)^i a^{i(m+2)}}{[(m+3)(m+6)-4][(2m+5)(2m+8)-4]\ \ldots\ [(im+1+2i)(im+4+2i)-4]}\right\}, \tag{42}$$

c being an unimportant integration constant. This series (as well as that of (39)) converges very rapidly for large values of m, but the convergence becomes bad as soon as m approaches -2.

Now, with respect to (4),

$$\frac{d\phi}{da} = \frac{\phi}{\psi}\left\{\frac{d\psi}{da} - a^2\phi\rho\right\}. \tag{43}$$

The reader will notice that the second right-hand term of (43) will vanish at the boundary, and so it follows from (4) and (43) that

$$\frac{1}{\phi_1}\left(\frac{d\phi}{da}\right)_1 = \frac{1}{\psi_1}\left(\frac{d\psi}{da}\right)_1, \tag{44}$$

i.e. that the ratios $\frac{\phi'}{\phi}$ and $\frac{\psi'}{\psi}$ at the boundary become identical. Their values for most values of m can easily be evaluated, using a few first terms of the developments in (42), and so we finally arrive at the values C defined by (2.1).

The numerical results (correct to four decimals) for nine values of m ranging from $-1{\cdot}5$ to 20 are given in Table I.

TABLE I

m	k	$\frac{\rho_c}{\rho_m}$	$\left(\frac{d\rho}{da}\right)_1$	$\frac{\psi_1}{c}$	$\frac{\psi_1'}{c}$	$\left(\frac{\phi'}{\phi}\right)_1$	C
20	8·3909	1·1412	20·5653	0·8854	0·2095	0·2366	1·6767
10	6·3507	1·2690	10·5943	0·8135	0·3426	0·4211	1·5489
5	5·0244	1·4785	5·6910	0·7267	0·4498	0·6189	1·4318
3	4·3776	1·7044	3·7478	0·6627	0·5168	0·7799	1·3490
2	4·0155	1·9185	2·7977	0·6167	0·5542	0·8987	1·2937
1	3·6081	2·3171	1·8727	0·5542	0·5858	1·0563	1·2270
0	3·1416	3·2899	1·0000	0·4838	0·6241	1·2899	1·1399
−1	2·5680	8·3142	0·2644	0·2983	0·4965	1·6643	1·0234
−1·5	2·0624	17·105	0·0829	0·1889	0·3946	2·0894	0·9170

Before discussing the results let us investigate our third model.

8. *Third Model.*—Let us turn back to (3) written in a slightly different form :

$$\frac{a^2}{6}\frac{d^2\phi}{da^2} + \frac{1}{3}\frac{\rho a^3}{\int_0^a \rho a^2 da}\left(a\frac{d\phi}{da} + \phi\right) = \phi, \tag{45}$$

and suppose that *

$$\frac{3}{a^3}\int_0^a \rho a^2 da = \rho_0(1-a)^\mu, \tag{46}$$

μ being a positive arbitrary parameter. The density distribution in a model defined in this way is

$$\rho = \rho_0(1-a)^{\mu-1}\left[1 - \left(1+\frac{\mu}{3}\right)a\right]. \tag{47}$$

As one remarks immediately, eq. (47) has one real zero solution within the range $0 < a < 1$, the boundary being now given by

$$a_1 = \frac{3}{3+\mu}, \tag{48}$$

* *Cf.* again Tisserand, *loc. cit.*, p. 138.

instead of $a_1 = 1$ as in the preceding case. The density condensation is clearly given by

$$\frac{\rho_c}{\rho_m} = \left(1 + \frac{3}{\mu}\right)^{\mu}. \tag{49}$$

With the aid of (46), eq. (45) can now be reduced to

$$a(1-a)\frac{d^2\phi}{da^2} + [6 - (\alpha+\beta+1)a]\frac{d\phi}{da} - \alpha\beta \,.\, a = 0, \tag{50}$$

where

$$\left.\begin{aligned} 2\alpha &= 5 + 2\mu + \sqrt{4\mu^2 + 12\mu + 25}, \\ 2\beta &= 5 + 2\mu - \sqrt{4\mu^2 + 12\mu + 25}. \end{aligned}\right\} \tag{51}$$

The solution of (50) can be expressed as follows :—

$$\phi = 1 + \frac{5!}{\Gamma(\alpha)\Gamma(\beta)} \sum_{i=1}^{\infty} \frac{\Gamma(\alpha+i)\Gamma(\beta+i)}{i!\,(i+5)!} a^i. \tag{52}$$

In one case this solution assumes a very simple form : that is for $\mu = 0$ (homogeneous model). In this case all terms behind the summation sign vanish ; we have

$$\phi = 1,$$

and consequently

$$\frac{1}{\phi}\frac{d\phi}{da} = 0, \tag{53}$$

in agreement with the preceding results. For any other value of μ, however, (52) becomes an infinite series. It can be generally shown * that such a series converges for all values of α and β, provided that $a < 1$ (which in consequence of (48) is actually the case), but, in practice, this convergence is very slow and renders the use of (52) highly impracticable. It is thus of great importance that by a suitable substitution it is possible (though only for certain distinct values of μ) to express the solution of (50) in a finite number of terms.

Let us change the variable and develop the solution of (50) in powers of $(1-a)^{-1}$ instead of a. It assumes the form : †

$$\phi = (1-a)^{-\beta}\left\{1 + \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(6-\alpha)} \sum_{i=1}^{\infty} \frac{\Gamma(\beta+i)\Gamma(6-\alpha+i)}{i!\,\Gamma(\beta-\alpha+i+1)} (1-a)^{-i}\right\}. \tag{54}$$

Provided that

$$\alpha = 5 + n, \tag{55}$$

n being a positive integer,‡ one easily verifies that the infinite series from (54) reduces to polynomials, and

$$\phi = (1-a)^{-\beta} G_n(\beta, a), \tag{56}$$

* See Bromwich, *An Introduction to the Theory of Infinite Series*, pp. 35, 37.

† See Forsyth, *loc. cit.*, pp. 207–216 ; *cf.* also Whittaker and Watson, *Modern Analysis*, pp. 281–286.

‡ The case $n = 0$ corresponds to $\mu = 0$, (54) being for $\mu = 0$ of course indefinite ; the reasons are obvious.

where *

$$G_n(\beta, a) = 1 + \frac{\Gamma(\beta-4-n)}{\Gamma(\beta)} \sum_{i=1}^{|n-1|} (-1)^i \binom{n-1}{i} \frac{\Gamma(\beta+i)}{\Gamma(\beta-4-n+i)} (1-a)^{-i}. \quad (57)$$

Now, at the boundary,

$$\left(\frac{\phi'}{\phi}\right)_{a=a_1} = \frac{\beta}{1-a_1} + \frac{G_n'(\beta, a_1)}{G_n(\beta, a_1)}, \quad (58)$$

and as from (48) and (51) it follows that

$$a_1 = \frac{6(1-\beta)}{(2-\beta)(3+\beta)}, \quad (59)$$

the boundary values of $G_n(\beta, a)$ and of its derivative are respectively :

$$G_n(\beta, a_1) = 1 + \frac{\Gamma(\beta-4-n)}{\Gamma(\beta+1)} \sum_{i=1}^{|n-1|} (-1)^i \binom{n-1}{i} \frac{(2-\beta)^i(3+\beta)^i\Gamma(\beta+i)}{\beta^{i-1}\Gamma(\beta-4-n+i)(5-\beta)^i}, \quad (60)$$

and

$$G_n'(\beta, a_1) = \frac{\Gamma(\beta-4-n)}{\Gamma(\beta+1)} \sum_{i=1}^{|n-1|} (-1)^i \binom{n-1}{i} \frac{(2-\beta)^{i+1}(3+\beta)^{i+1}\Gamma(\beta+i)}{\beta^i\Gamma(\beta-4-n+i)(5-\beta)^{i+1}}. \quad (61)$$

Table II contains the results of numerical computations for five values of n ranging from 1 to 10 ; the C's have been evaluated with the aid of (2.1) and (58) in the usual way.

TABLE II

n	α	β	μ	a_1	$\frac{\rho_c}{\rho_m}$	G_n	G_n'	$\left(\frac{\phi'}{\phi}\right)_{a=a_1}$	C
1	6	0·2000	0·6000	0·8333	2·9301	1·0000	0·0000	1·2000	1·2501
2	7	0·3333	1·1667	0·7200	4·4157	1·2101	0·7503	1·8105	1·1351
4	9	0·5000	2·2500	0·5417	6·7287	1·3042	1·7386	2·4998	1·1180
6	11	0·6000	3·3000	0·4762	8·4477	1·3557	2·1326	2·7186	1·1381
10	15	0·7143	5·3171	0·3590	10·8297	1·3981	2·4611	2·9051	1·2324

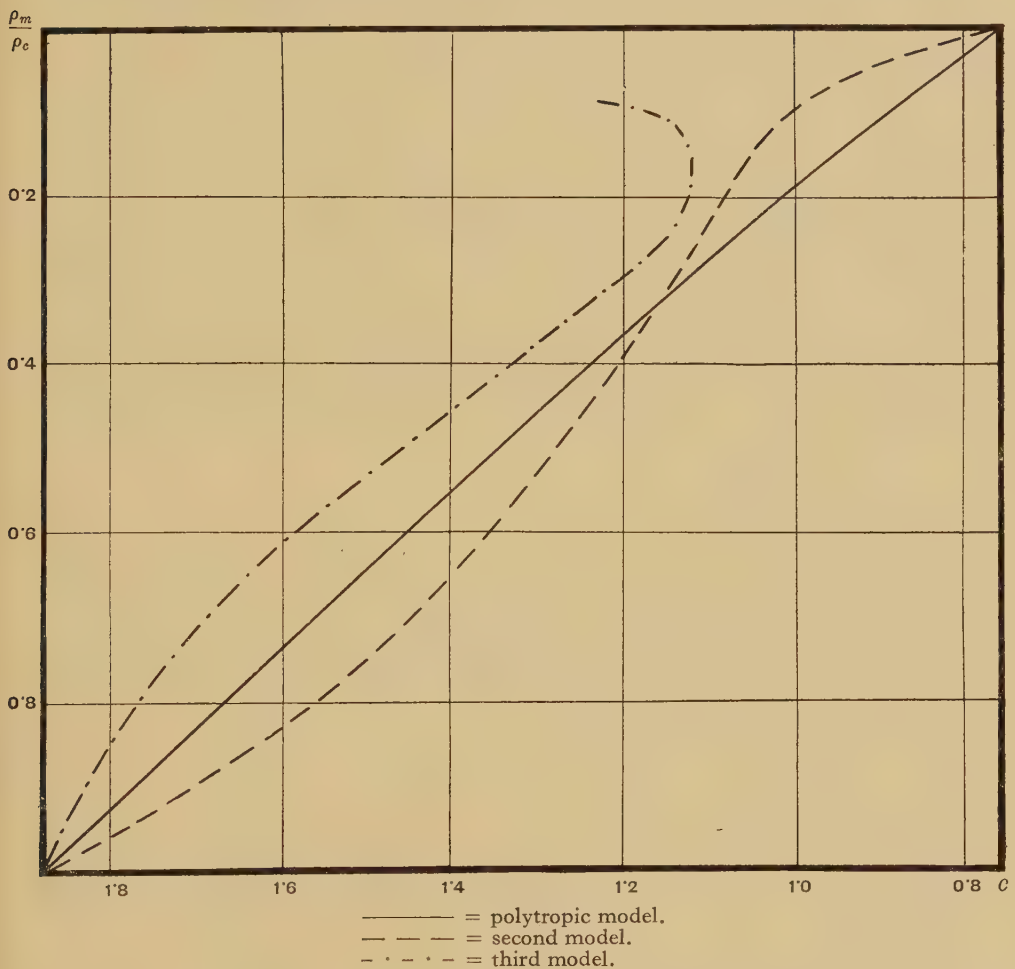
In order to be able to draw unambiguous conclusions as to the dependence of C on $\frac{\rho_c}{\rho_m}$, it was found necessary to ascertain the C's for some value of μ lying within the interval $0 < \mu < \frac{3}{5}$. The numerical calculations directly using (52) have been carried out for $\mu = \frac{1}{20}$ and $\frac{1}{5}$, and the results are :

μ	α	β	$\frac{\rho_c}{\rho_m}$	a_1	$\left(\frac{\phi'}{\phi}\right)_{a=a_1}$	C
0·05	5·080	0·0197	1·228	0·984	0·114	1·775
0·2	5·325	0·0751	1·741	0·938	0·460	1·542

9. *Conclusion.*—The complete march of the dependence of C on the density condensation for the three discussed models is diagramatically

* By an obvious extension of an argument given in Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. i, pp. 76–77.

shown in the figure.* We see that the constant of proportionality between e and v does, in fact, depend on the kind of model adopted, being a function of the density condensation as well as of the whole march of the density



distribution. The extent and character of this dependence is readily found to be different for different models.

But, as one easily verifies, in the case of all three models the constant C is very sensitive to any change of the density condensation for $\frac{\rho_c}{\rho_m}$ not too far from unity, but the sensitivity rapidly decreases with increase of $\frac{\rho_c}{\rho_m}$. Thus, from the general nature of our diagram, we easily realise that for small values of C (*i.e.* values near to $\frac{3}{4}$) the problem of determining the density condensations from C remains, in practice, indeterminate in

* The curve for the polytropic model has been drawn according to the investigations of Chandrasekhar (*M.N.*, **93**, 553, 1933, Table I).

so far as we do not know the law which governs the internal density distribution, since at best only crude estimations can be made in this way. For instance, if $C=\frac{4}{5}$, in the case of a polytrope there would result $\frac{\rho_c}{\rho_m}=29$, but the alternative values are 67 and even greater, according to whether the configuration follows the model II or III respectively.

But, on the other hand, for larger values of C the feature of the problem changes. In some measure as C approaches its upper limit, the differences between various models become less and less important, and so C becomes to a large extent characteristic for the density condensation alone.*

This is obviously of great importance, because in the underlying astronomical problem we are chiefly concerned with such values of C as fall into the last category. Thus, having in mind our astronomical problem, we may summarise as follows : the values of density condensations arrived at in the case of eclipsing binaries from the study of their ellipticities, using the polytropic model, are true (as regards order of magnitude at any rate), no matter whether real stars are complete polytropes or not.

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1937 *October.*

* This has been found true for the three models discussed, and will probably be true for all models possessing a negative density gradient throughout. But the question still remains open as to the behaviour of models with a reversal of the density gradient, which may well be of astrophysical importance and on which our present analysis has obviously no bearing.

MEAN AREAS AND HELIOGRAPHIC LATITUDES OF SUNSPOTS IN THE YEAR 1936.

Royal Observatory, Greenwich.

(Communicated by the Astronomer Royal)

The following results are in continuation of those given in *M.N.*, **97**, 76, 1936, and are derived from the measurement at Greenwich of photographs taken at the Royal Observatories of Greenwich and the Cape and at the Kodaikanal Observatory, India.

Table I gives the mean daily areas of umbræ, whole spots and faculæ for each synodic rotation of the Sun included in the year 1936; the means for the year as a whole are included in Table II, which summarizes the yearly values from 1933–36.

TABLE I

No. of Rotation	Rotation Commenced U.T.	Days Photographed	Mean Daily Areas: Projected *			Mean Daily Areas: Corrected for Foreshortening †		
			Umbræ	Whole Spots	Faculæ	Umbræ	Whole Spots	Faculæ
1101	1936 Jan. 2·74	27	380	2315	2101	285	1714	2481
1102	Jan. 30·08	27	307	1514	1961	240	1178	2410
1103	Feb. 26·42	28	291	1406	2169	232	1130	2583
1104	Mar. 24·74	27	321	1697	2140	236	1231	2600
1105	Apr. 21·01	27	154	789	2078	115	578	2546
1106	May 18·25	28	170	839	1600	123	612	1965
1107	June 14·45	27	203	1047	1718	169	873	2097
1108	July 11·65	27	195	972	1671	145	706	2006
1109	Aug. 7·86	27	246	1227	2110	181	895	2554
1110	Sept. 4·11	27	230	1130	2035	186	943	2456
1111	Oct. 1·38	28	293	1502	2439	214	1092	3011
1112	Oct. 28·67	27	380	2063	2209	272	1453	2643
1113	Nov. 24·98	27	505	3060	2784	372	2201	3307

TABLE II

Year	No. of Days: Photographed	No. of Days: Without Spots	Mean Daily Areas: Projected *			Mean Daily Areas: Corrected for Foreshortening †		
			Umbræ	Whole Spots	Faculæ	Umbræ	Whole Spots	Faculæ
1933	365	234	24	120	225	18	88	267
1934	365	145	27	142	297	22	119	354
1935	365	21	152	810	910	117	624	1100
1936	366	0	288	1529	2106	217	1141	2545

* Expressed in millionths of the Sun's disc.

† Expressed in millionths of the Sun's hemisphere.

Table III gives for each rotation in the year 1936 the mean daily area of the whole spots (corrected for foreshortening) and the mean heliographic latitude of the spotted areas for both northern and southern hemispheres.

TABLE III

No. of Rotation	Rotation Commenced U.T.	Spots North of the Equator		Spots South of the Equator		Mean Latitude of entire Spotted Area	Mean Distance from Equator of all Spots
		Mean Daily Area	Mean Heliographic Latitude	Mean Daily Area	Mean Heliographic Latitude		
			°		°	°	°
1101	1936 Jan. 2·74	557	23·15	1157	25·23	− 9·51	24·56
1102	Jan. 30·08	393	20·83	785	25·78	−10·23	24·13
1103	Feb. 26·42	373	21·22	758	23·51	− 8·76	22·76
1104	Mar. 24·74	454	17·95	777	21·82	− 7·14	20·39
1105	Apr. 21·01	84	19·39	494	22·48	−16·39	22·03
1106	May 18·25	111	16·31	501	20·80	−14·07	19·98
1107	June 14·45	625	20·33	248	18·15	+ 9·39	19·71
1108	July 11·65	383	22·02	324	20·37	+ 2·60	21·26
1109	Aug. 7·86	395	24·70	500	24·06	− 2·55	24·34
1110	Sept. 4·11	306	14·55	637	21·05	− 9·50	18·94
1111	Oct. 1·38	573	18·30	519	16·37	+ 1·83	17·38
1112	Oct. 28·67	443	15·34	1011	14·68	− 5·53	14·88
1113	Nov. 24·98	1046	17·22	1155	17·97	− 1·25	17·62

The mean heliographic latitude of the entire spotted area and the mean distance from the equator of all spots are also tabulated. The mean value for the year 1936 is included in Table IV.

TABLE IV

Year	Days Photographed	Spots North of the Equator		Spots South of the Equator		Mean Latitude of entire Spotted Area	Mean Distance from Equator of all Spots
		Mean Daily Area	Mean Heliographic Latitude	Mean Daily Area	Mean Heliographic Latitude		
			°		°	°	°
1933	365	86	10·60	2	8·35	+10·23	10·56
1934	365	44	16·45	74	28·09	−11·49	23·75
1934	Old Cycle	19	4·10	1	4·37	+ 3·79	4·11
	New Cycle	26	25·44	74	28·31	−14·46	27·57
1935	365	205	22·42	419	23·72	− 8·57	23·30
1936	366	463	19·36	678	21·02	− 4·63	20·35

Tables II and IV are in continuation of similar tables given in *Monthly Notices*: for the years 1874 to 1888—**49**, 381–382; 1889 to 1902—**63**, 465–466; 1901 to 1914—**76**, 402–403; 1913 to 1924—**85**, 1007–1008; 1923 to 1933—**94**, 870–871.

The rotations in Tables I and III are numbered in continuation of Carrington's series.

The chief features of the record of sunspots and faculæ for the year 1936 are as follows :—

1. A continuation of the steep rise towards the maximum of the 11-year cycle, the ratio of the mean daily areas (corrected for foreshortening) for 1935 and 1936 being 1 : 1·8 for sunspots and 1 : 2·3 for faculæ. There were 14 large groups of spots, each exceeding 500 millionths of the Sun's hemisphere in mean area. One of these groups, with central meridian passage on January 19·9, had a mean area of 1153 millionths.

2. On no day in the year 1936 were sunspots and faculæ absent from the disc. This is the first record of no spotless days in any one year since the three years 1927–1929. At the last minimum there were 234 spotless days in 1933.

3. The respective mean daily areas of sunspots in the southern hemisphere and in the northern were in the ratio of about 3 : 2 (in 1935 the corresponding ratio was 2 : 1). This preponderance of activity in the southern hemisphere is due to the extra number therein of moderate-sized groups, of area 100 to 400 millionths of the Sun's hemisphere, the very small groups and the largest groups (as a whole) being slightly more numerous in the northern hemisphere.

4. The number of groups lasting for two days or longer was 379, of which 184 were in the northern hemisphere and 195 in the southern. In addition, there were 145 very small groups recorded on one day only, of which 75 were north and 70 were south of the equator.

5. The mean latitude of all spots irrespective of sign is 20°·35, *i.e.* 3° nearer the equator than in 1935. The number of spots nearer the equator than ± 10° was 4 per cent. of the total number of all groups recorded during the year. The magnetic polarities of these equatorial groups appear, from the published Mount Wilson observations, to be consistent, as regards sign, with the scheme of polarities holding for the present 11-year cycle.

Royal Observatory, Greenwich :
1938 *February* 21.

OCCULTATIONS OF STARS BY THE MOON OBSERVED AT THE UNIVERSITY OF LONDON OBSERVATORY

Frank Robbins.

This report concerning occultations observed and reduced at Mill Hill is the seventh of a series of which No. 6 is in *M.N.*, **97**, 546,1937. Each of the thirty-eight occultations (January–June) predicted in the *Nautical Almanac* was attempted, but owing to clouds only six were observed, and these with one unpredicted make a scanty list of seven.

In each case the 8-inch Cooke refractor with a Ramsden ocular magnifying 130 times was used, and by the eye and ear method.

Column 6 of Table I contains the actual U.T. of the observations. For the purpose of interpolating for parallax, right ascension and declination of the Moon (but for no other purpose) $0^h.00152$ was added to the U.T. as desired by Emeritus Professor E. W. Brown.

No correction for lag, personal equation or for contour irregularities on the limb of the Moon, etc., has been made. It has been my custom to reduce each observation on the following morning by the method devised by Innes of Johannesburg. These reductions were themselves checked in December by repeating the reduction using the modification due to Dr. L. J. Comrie.

While it is not to be expected that the two processes will yield *precisely* the same result, it is of interest to note that in finding $(\sigma' - \sigma)$ the greatest discrepancy was $0''.2$. For the error of the synchronome clock I relied upon the six-dot time-signals broadcast by the B.B.C. both before and after each occultation.

Finally, I must express my gratitude to the Director of University of London Observatory for his kindness in affording me facilities with regard to the 8-inch refractor during six and a half happy years.

TABLE I

No.	Date	Star	Mag.	Phase	U.T.
	1937				h m s
160	Jan. 2	55 Leonis	6·0	R	3 46 32·36
161	Feb. 17	B.D. +20° 573	7·2	D	19 55 29·62
162	Mar. 18	108 Tauri	6·2	D	19 38 27·90
163	19	B.D. +21° 1203	7·1	D	22 15 20·93
164	19	B.D. +21° 1204	7·7	D	22 6 23·60
165	May 16	ω Leonis	5·5	D	21 5 57·56
166	19	13 B Virginis	5·8	D	21 20 10·66

TABLE II

No.	χ	ρ	$\chi - \rho$	$\cos(\chi - \rho)$	$\sin(\chi - \rho)$	$\sigma' - \sigma$
	° ′	° ′	° ′			″
160	337 14	113 26	223 48	−·722	−·692	+1·5
161	138 27	80 10	58 17	+·526	+·851	+2·0
162	156 24	90 22	66 2	+·406	+·914	−0·2
163	165 35	96 47	68 48	+·362	+·932	+0·3
164	141 10	96 45	44 25	+·714	+·700	+1·5
165	107 2	110 24	356 38	+·998	−·059	−1·2
166	87 19	112 16	335 3	+·907	−·422	−1·8

TABLE III

No.	R.A. 1937·0	Red. to date	Dec. 1937·0	Red. to date	Authority
	h m s	s	° ′ ″	″	
163	6 17 28·31	+2·12	+21 9 44·0	−2·7	*N.A.*
164	6 17 35·91	+2·13	+21 13 47·4	−2·6	Abbadia, +16° to +24°

Erratum to "The System of α Ursæ Majoris" by H. Spencer Jones and H. H. Furner.

In Table VIII, *Ephemeris*, the quantities $\Delta\alpha$, $\Delta\delta$ by an oversight refer to the years 1947·0 to 1957·0 and, as tabulated, are corrections to be applied to observed positions to give the centre of gravity. The following values should be substituted to give $\Delta\alpha$, $\Delta\delta$ as defined in § VIII :—

	$\Delta\alpha$	$\Delta\delta$		$\Delta\alpha$	$\Delta\delta$
	s	″		s	″
1937·0	+·028	−·17	1943·0	+·031	−·07
38·0	+·029	−·16	44·0	+·031	−·05
39·0	+·030	−·14	45·0	+·030	−·03
40·0	+·030	−·13	46·0	+·029	−·01
41·0	+·031	−·11	47·0	+·028	+·02
42·0	+·031	−·09			